Numerical Solution of Equation of Motion

Average Acceleration Method (Trapezoidal method)

$m\,a + c\,v + k\,d = f\,(t)$

In the above mass-spring-dashpot (damper) equation of motion, f (t) is the forcing function defined as given f_i values at (t_i) in which i = 0, 1, 2, … NTS (number of time steps), with the time step between any t_i and t_{i+1} equal to Δt, and m, c, k are the mass, damping and stiffness coefficients.

Initial Conditions: $d\,(t = 0) = d_0$, and $v\,(t = 0) = v_0$

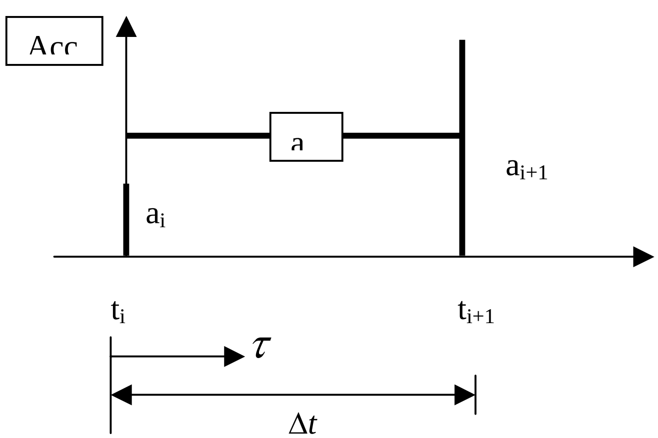
From these conditions and the known f_0, you can find $a\,(t = 0) = a_0$ from the Equation above

At any time step $t = t_{i+1}$: $m\,a_{i+1} + c\,v_{i+1} + k\,d_{i+1} = f_{i+1}$ (Eq. 1)

Now, we need to find a_{i+1}, v_{i+1}, and d_{i+1}, using f_{i+1}, and information from the previous time step (i.e., a_i, v_i, and d_i)

Average acceleration dictates that (see figure):

$a = (a_{i+1} + a_i) / 2$ (Eq. 2)



Integrate to get velocity

$v = v_i + \tau (a_{i+1} + a_i) / 2$ (Eq. 3)

Integrate above to get displacement:

$d = d_i + v_i \tau + (\tau^2 /4) (a_{i+1} + a_i)$ (Eq. 4)

At the end of the Interval, $\tau = \Delta t$ and therefore (using Eqs. 3 and 4)

$v_{i+1} = v_i + (\Delta t/ 2) (a_{i+1} + a_i)$ (Eq. 5)

$d_{i+1} = d_i + v_i \Delta t + (\Delta t^2 / 4) (a_{i+1} + a_i)$ (Eq. 6)

Now, substitute Equations 5 and 6 into Equation 1 to get

$m a_{i+1} + c (v_i + (\Delta t/ 2) (a_{i+1} + a_i)) + k (d_i + v_i \Delta t + (\Delta t^2 / 4) (a_{i+1} + a_i)) = f_{i+1}$

or,

$(m + c (\Delta t / 2) + k (\Delta t^2 / 4)) a_{i+1} = f_{i+1} - c (v_i + (\Delta t/ 2) a_i) - k (d_i + v_i \Delta t + (\Delta t^2 / 4)a_i)$ (Eq. 7)

(where $[m + c (\Delta t / 2) + k (\Delta t^2 / 4)]$ is known as the effective mass = m*)

Solve Eq. 7 for a_{i+1}, and (using Eqs. 4 and 5) solve for v_{i+1} , d_{i+1}

Now all quantities are known at i +1 and we are ready to go to the next time step (repeat the above procedure).

Summary for numerical implementation

Define m, c, k, d_0, v_0, NTS, f (NTS), Δt
(note: the first stored value in the array f is f_0)

Declare arrays a(NTS), v(NTS), d(NTS), t(NTS)
notes: 1. NTS is the number of time steps or the number of data points that constitute (define) the forcing function f.
2. The arrays above for a, v, d, and t are declared now, but will be defined as the calculations proceed.

Find $a_0 = (f_0 - c v_0 - k d_0) / m$

Define $t_0 = 0.0$

Define the first entries in arrays a, v, d, and t to be a_0, v_0, d_0, and t_0, respectively.

Define the effective mass

$m^* = [m + c (\Delta t / 2) + k (\Delta t^2 / 4)]$

Set i = 0

Do while i less than NTS

$a_{i+1} = \{f_{i+1} – c (v_i + (\Delta t/ 2) a_i) - k (d_i + v_i \Delta t + (\Delta t^2 / 4) a_i)\}/ (m + c (\Delta t / 2) + k (\Delta t^2 / 4))$

$v_{i+1} = v_i + (\Delta t/ 2) (a_{i+1} + a_i)$

$d_{i+1} = d_i + v_i \Delta t + (\Delta t^2 / 4) (a_{i+1} + a_i)$

$t_{i+1} = t_i + \Delta t$

Store $a_{i+1,}$ $v_{i+1,}$ $d_{i+1,}$ and t_{i+1} in next row of a, v, d, and t arrays respectively.

Define i = i + 1

Continue

Now you have defined arrays a(NTS), v(NTS), d(NTS) and t(NTS) for plotting if you wish..

Predictor Multi-corrector Implementation

Case of nonlinear Elastic-Perfectly Plastic Restoring Force

We know that:

$v_{i+1} = v_i + (\Delta t/ 2) (a_{i+1} + a_i)$

$d_{i+1} = d_i + v_i \Delta t + (\Delta t^2 / 4) (a_{i+1} + a_i)$

The equation is:

$m\ a + c\ v + fs = f\ (t)$

Define m, c, k, d_0, v_0, fmax, fmin, NTS, f (NTS), Δt
(note: the first stored value in the array f is f_0)

Declare arrays a(NTS), v(NTS), d(NTS), fs(NTS), t(NTS)

notes: 1. NTS is the number of time steps or the number of data points that constitute (define) the forcing function f.

2. The arrays above for a, v, d, fs, and t are declared now, but will be defined as the calculations proceed.

$fs_0 = k\ d_0$

If fs_0 greater than fmax, fs_0 = fmax

If fs_0 less than fmin, fs_0 = fmin

Find $a_0 = (f_0 - c\ v_0 - fs_0) / m$

Define $t_0 = 0.0$

Define the first entries in arrays a, v, d, fs, and t to be a_0, v_0, d_0, fs_0, and t_0, respectively.

Set i = 0

Do while i less than NTS

1. Start with i = 0

At $t = t_{i+1}$: $m\ a_{i+1} + c\ v_{i+1} + fs_{i+1} = f_{i+1}$

Now, we need to find a_{i+1}, v_{i+1}, and d_{i+1}, using f_{i+1}, and information from the previous time step (i.e., a_i, v_i, and d_i)

2. Predictor Phase (start with the known components of v_{i+1} and d_{i+1}, i.e., all terms except those involving a_{i+1})

$v^p_{i+1} = v_i + (\Delta t/ 2)\ a_i$

$d^p_{i+1} = d_i + v_i\ \Delta t + (\Delta t^2 / 4)\ a_i$

$v_{i+1} = v^p_{i+1}$

$d_{i+1} = d^p_{i+1}$

$a_{i+1} = 0$

Set iteration number n to zero (n = 0)

3. Find residual force Δf and tangent stiffness k_t

$k_t = k$
$fs_{i+1} = fs_i + k_t\,(d_{i+1} – d_i)$
if fs_{i+1} greater than fmax, fs_{i+1} = fmax and $k_t = 0$
if fs_{i+1} less than fmin, fs_{i+1} = fmin and $k_t = 0$

$\Delta f = f_{i+1} - m\, a_{i+1} - c\, v_{i+1} - fs_{i+1}$

If n = 0 , $\Delta f0 = \Delta f$

Check convergence tolerance (i.e., Δf is nearly zero already)
Tol = $(| \Delta f | / | \Delta f0 |)$ (in which | | denotes absolute value)

If Tol less than or equal 10^{-5} (or similar small number) go to 6

n = n + 1

Form effective mass m*

$m^* = [m + c\,(\Delta t / 2) + k_t\,(\Delta t^2 / 4)]$

4. Solve $m^*\, \Delta a_{i+1} = \Delta f$ for Δa_{i+1}

5. Corrector phase (add the remaining terms)

$a_{i+1} = a_{i+1} + \Delta a_{i+1}$

$v_{i+1} = v^p_{i+1} + (\Delta t/2)\, a_{i+1}$

$d_{i+1} = d^p_{i+1} + (\Delta t^2/4)\, a_{i+1}$

Go to step 3

6. Now all quantities are known at i +1. Store a_{i+1}, v_{i+1}, d_{i+1}, fs_{i+1}, and t_{i+1} in next row of a, v, d, fs, and t arrays respectively. We are ready to go to the next time step (i.e., i = i + 1, and go to 1).

Numerical Solution of Equation of Motion

Linear Acceleration Method

$m\,a + c\,v + k\,d = f\,(t)$

In the above mass-spring-dashpot (damper) equation of motion, f (t) is the forcing function defined as given f_i values at (t_i) in which i = 0, 1, 2, … NTS (number of time steps), with the time step between any t_i and t_{i+1} equal to Δt, and m, c, k are the mass, damping and stiffness coefficients.

Initial Conditions: $d\,(t = 0) = d_0$, and $v\,(t = 0) = v_0$

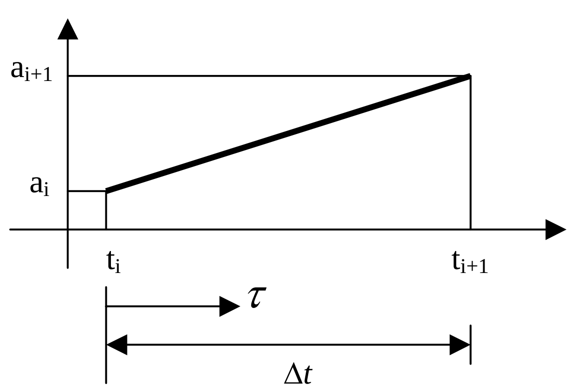
From these conditions and the known f_0, you can find $a\,(t = 0) = a_0$ from the Equation above

At any time step $t = t_{i+1}$: $m\,a_{i+1} + c\,v_{i+1} + k\,d_{i+1} = f_{i+1}$ (Eq. 1.1)

Now, we need to find a_{i+1}, v_{i+1}, and d_{i+1}, using f_{i+1}, and information from the previous time step (i.e., a_i, v_i, and d_i)

Linear acceleration dictates that (see figure):

$a = a_i + (\tau / \Delta t)\,(a_{i+1} - a_i)$



Integrate to get velocity

$v = a_i\,\tau + (\tau^2 / 2\Delta t)\,(a_{i+1} - a_i) + c1$ (c1 is a constant)

At $\tau = 0$, $v = v_i$ (therefore c1 above is equal to v_i), or

$$v = a_i \tau + (\tau^2 / 2\Delta t)(a_{i+1} - a_i) + v_i \quad \text{(Eq. 1.2)}$$

Integrate above to get expression for displacement:

$$d = a_i \tau^2 /2 + (\tau^3 / 6\Delta t)(a_{i+1} - a_i) + v_i \tau + c1$$

At $\tau = 0$, $d = d_i$ (therefore c1 above is equal to d_i), or

$$d = a_i \tau^2 /2 + (\tau^3 / 6\Delta t)(a_{i+1} - a_i) + v_i \tau + d_i \quad \text{(Eq. 1.3)}$$

At the end of the Interval, $\tau = \Delta t$ and therefore (using Eqs. 1.2 and 1.3)

$$v_{i+1} = a_i \Delta t + (\Delta t / 2)(a_{i+1} - a_i) + v_i$$

or

$$v_{i+1} = v_i + (\Delta t / 2)(a_{i+1} + a_i) \quad \text{(Eq. 1.4)}$$

and

$$d_{i+1} = d_i + v_i \Delta t + (\Delta t^2 / 6) a_{i+1} + (\Delta t^2 / 3) a_i \quad \text{(Eq. 1.5)}$$

Now, substitute Equations 1.4 and 1.5 into Equation 1.1 to get

$$m a_{i+1} + c(v_i + (\Delta t/2)(a_{i+1} + a_i)) + k(d_i + v_i \Delta t + (\Delta t^2/6) a_{i+1} + (\Delta t^2/3) a_i) = f_{i+1} \quad \text{(Eq. 1.6)}$$

or,

$$(m + c(\Delta t / 2) + k(\Delta t^2 / 6)) a_{i+1} = f_{i+1} - c(v_i + (\Delta t/2) a_i) - k(d_i + v_i \Delta t + (\Delta t^2 / 3) a_i) \quad \text{(Eq. 1.7)}$$

(where $[m + c(\Delta t / 2) + k(\Delta t^2 / 6)]$ is known as the effective mass = m*)

Solve Eq. 1.7 for a_{i+1}, and (using Eqs. 1.4 and 1.5) solve for v_{i+1} , d_{i+1}

Now all quantities are known at i +1 and we are ready to go to the next time step (repeat the above procedure).

Summary for numerical implementation

Define m, c, k, d_0, v_0, NTS, f (NTS), Δt
(note: the first stored value in the array f is f_0)

Declare arrays a(NTS), v(NTS), d(NTS), t(NTS)
notes: 1. NTS is the number of time steps or the number of data points that constitute (define) the forcing function f.
2. The arrays above for a, v, d, and t are declared now, but will be defined as the calculations proceed.

Find $a_0 = (f_0 - c\, v_0 - k\, d_0) / m$

Define $t_0 = 0.0$

Define the first entries in arrays a, v, d, and t to be a_0, v_0, d_0, and t_0, respectively.

Define the effective mass

$m^* = [m + c\,(\Delta t / 2) + k\,(\Delta t^2 / 6)]$

Set i = 0

Do while i less than NTS

$a_{i+1} = \{f_{i+1} - c\,(v_i + (\Delta t/2)\, a_i) - k\,(d_i + v_i\,\Delta t + (\Delta t^2 / 3)\, a_i)\} / (m + c\,(\Delta t / 2) + k\,(\Delta t^2 / 6))$

$v_{i+1} = v_i + (\Delta t/2)\,(a_{i+1} + a_i)$

$d_{i+1} = d_i + v_i\,\Delta t + (\Delta t^2 / 6)\, a_{i+1} + (\Delta t^2 / 3)\, a_i$

$t_{i+1} = t_i + \Delta t$

Store a_{i+1}, v_{i+1}, d_{i+1}, and t_{i+1} in next row of a, v, d, and t arrays respectively.

Define i = i + 1

Continue

Now you have defined arrays a(NTS), v(NTS), d(NTS) and t(NTS) for plotting if you wish..

Predictor Multi-corrector Implementation

Nonlinear Elastic-Perfectly Plastic Restoring Force

We know that:

$v_{i+1} = v_i + (\Delta t/ 2) (a_{i+1} + a_i)$

$d_{i+1} = d_i + v_i \Delta t + (\Delta t^2 / 6) a_{i+1} + (\Delta t^2 / 3) a_i$

The equation is:

$m a + c v + fs = f (t)$

Define m, c, k, d_0, v_0, fmax, fmin, NTS, f (NTS), Δt
(note: the first stored value in the array f is f_0)

Declare arrays a(NTS), v(NTS), d(NTS), fs(NTS), t(NTS)
notes: 1. NTS is the number of time steps or the number of data points that constitute (define) the forcing function f.
2. The arrays above for a, v, d, fs, and t are declared now, but will be defined as the calculations proceed.

$fs_0 = k d_0$
If fs_0 greater than fmax, fs_0 = fmax
If fs_0 less than fmin, fs_0 = fmin
Find $a_0 = (f_0 - c v_0 - fs_0) / m$

Define $t_0 = 0.0$

Define the first entries in arrays a, v, d, fs, and t to be a_0, v_0, d_0, fs_0, and t_0, respectively.

Set i = 0

Do while i less than NTS

1. Start with i = 0

At $t = t_{i+1}$: $m a_{i+1} + c v_{i+1} + fs_{i+1} = f_{i+1}$

Now, we need to find a_{i+1}, v_{i+1}, and d_{i+1}, using f_{i+1}, and information from the previous time step (i.e., a_i, v_i, and d_i)

2. Predictor Phase (start with the known components of v_{i+1} and d_{i+1}, i.e., all terms except those involving a_{i+1})

$v^p_{i+1} = v_i + (\Delta t/ 2) a_i$

$d^p_{i+1} = d_i + v_i \Delta t + (\Delta t^2 / 3) a_i$

$v_{i+1} = v^p_{i+1}$

$d_{i+1} = d^p_{i+1}$

$a_{i+1} = 0$

Set iteration number n to zero (n = 0)

3. Find residual force Δf and tangent stiffness k_t

$k_t = k$
$fs_{i+1} = fs_i + k_t (d_{i+1} – d_i)$
if fs_{i+1} greater than fmax, fs_{i+1} = fmax and $k_t = 0$
if fs_{i+1} less than fmin, fs_{i+1} = fmin and $k_t = 0$

$\Delta f = f_{i+1} - m\, a_{i+1} - c\, v_{i+1} - fs_{i+1}$

If n = 0 , $\Delta f0 = \Delta f$

Check convergence tolerance (i.e., Δf is nearly zero already)
Tol = $(| \Delta f | / | \Delta f0 |)$

If Tol less than or equal 10^{-5} (or similar small number) go to 6

n = n + 1

Form effective mass m*

$m^* = [m + c (\Delta t / 2) + k_t (\Delta t^2 / 6)]$

4. Solve $m^* \Delta a_{i+1} = \Delta f$ for Δa_{i+1}

5. Corrector phase (add the remaining terms)

$a_{i+1} = a_{i+1} + \Delta a_{i+1}$

$v_{i+1} = v^p_{i+1} + (\Delta t/2)\, a_{i+1}$

$d_{i+1} = d^p_{i+1} + (\Delta t^2/6)\, a_{i+1}$

Go to 3

6. Now all quantities are known at i +1. Store $a_{i+1,}$ $v_{i+1,}$ $d_{i+1,}$ fs_{i+1}, and t_{i+1} in next row of a, v, d, fs, and t arrays respectively. We are ready to go to the next time step (i.e., i = i + 1, and go to 1).

Notes

1. For earthquake base excitation, f(t) is replaced by $-m\ddot{u}_g(t)$ where $\ddot{u}_g(t)$ is the time history of ground acceleration**. All units must conform to those of this input acceleration (e.g., m/sec/sec).** In this case, absolute acceleration is $\ddot{u}_{abs} = (a + \ddot{u}_g)$.

2. The above procedures apply to solutions of multi-degree of freedom (matrix) systems of equations. In this case, m, k, and c are replaced by M, K and C (NxN matrices, where N is number of degrees of freedom), and a, v, d, f, and so forth become vectors of size N. In the solution, a matrix inversion step will be needed, which can be conveniently performed using many readily available algorithms (please see any numerical analysis recipe book).

3. The above average and linear acceleration methods are special cases of a general procedure known as the Newmark time integration procedure (see Chopra for more details).

Optional Exercise (Use Average Acceleration Method)

(See Note 1 above)

1. Define m, k and c for a system of natural frequency ω (of your choice), and a damping of 5 %. Shake with ten cycles of a square wave **base excitation** (of your choice), and plot absolute acceleration and relative displacement (about 20 cycles), followed by 10 seconds of free vibration. Comment on the result (forced vibration phase, free vibration phase). Estimate damping from your answer using the logarithmic increment method (is it 5%?…, it should be). Check the natural period of your system as well.

(Use suggested values of: $m = 1.0$, $\omega = 4\pi$, square wave of 4 cycles per second, input amplitude $= 1.0$, and $\Delta t = 0.0125$ second). Hint: Add trailing zeros to your input excitation vector for the free vibration phase.

2. Define m, k and c, fmax, fmin for a nonlinear elastic-perfectly plastic system (of a linear natural frequency ω of your choice), and a damping of 5 %. Shake with ten cycles of a square wave **base excitation** (of your choice), followed by 10 seconds of free vibration. Plot absolute acceleration, relative displacement, and fs versus relative displacement (about 10 cycles). The shaking amplitude should be high enough to result in nonlinear response. Comment on the result (forced vibration phase, free vibration phase).

(Use suggested values of: $m = 1.0$, $\omega = 4\pi$, square wave of 4 cycles per second, input amplitude $= 1.0$, $\Delta t = 0.0125$ second, fmax = 0.5 of $ku_{(max)}$ from problem 1 above, and fmin = -fmax). Hint: Add trailing zeros to your input excitation vector for the free vibration phase.