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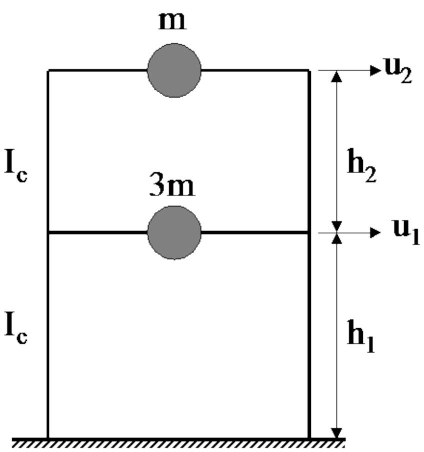
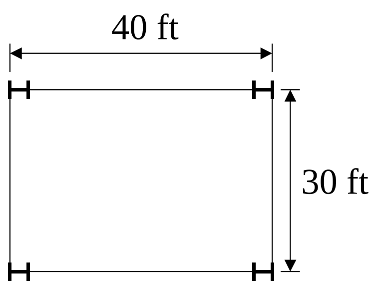
Solved MDOF and Modal Analysis Example



Elevation View



Plan View

Use:

$E_{steel} = 29{,}000$ ksi
$I_c = 164.8\ in^4$
$h_1 = 15$ ft
$h_2 = 12$ ft

$$m = \frac{W}{g} = \left(30\frac{lb}{ft^2}\right)\left(\frac{40\,ft \times 30\,ft}{386.4\,in/s^2}\right) = 93.17\frac{lb\cdot s^2}{in} = 0.09317\frac{kip\cdot s^2}{in}$$

$$k_{column} = \left(\frac{12EI_c}{h_i^3}\right) \Rightarrow k_{floor\ i} = 4\left(12\frac{EI_c}{h_i^3}\right) = \frac{48EI_c}{h_i^3}$$

(a) For the two-story building shown above, define the two-degree-of-freedom free vibration matrix equation in terms of k and m. Using this matrix equation, determine the natural frequencies ω_1 and ω_2 (in terms of k and m). Using these expressions for ω_1 and ω_2, plug in numerical values and determine ω_1 and ω_2 in radians. For each natural frequency, define and sketch the corresponding mode shape.

(b) Verify that the modes are orthogonal as expected.

(c) Normalize the first mode such that $\boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1 = 1.0$

(d) Use the normalized first mode (from above) to verify that $\boldsymbol{\phi}_1^T \mathbf{k} \boldsymbol{\phi}_1 = \omega_1^2$

(e) Use the El Centro Response Spectrum and a damping ratio of 5% to estimate the maximum base shear and moment.

(f) Find a_0 and a_1 in $\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k}$ for a viscous damping of 5% in modes 1 and 2.

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$$k_1 = \frac{48EI_c}{h_1^3} = \frac{48(29000\,ksi)(164.8\,in^4)}{(180\,in)^3} = 39.335\,kip/in$$

$$k_2 = \frac{48EI_c}{h_2^3} = \frac{48(29000\,ksi)(164.8\,in^4)}{(144\,in)^3} = 76.826\,kip/in$$

$$m = 0.09317\frac{kip \cdot s^2}{in}$$

Write the equation of motion in matrix form:

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

substitute in $\ddot{u} = -\omega^2 u$ and rearrange to get

$$\begin{bmatrix} k_1 + k_2 - \omega^2 3m & -k_2 \\ -k_2 & k_2 - \omega^2 m \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Take the determinant and set equal to zero

$$\begin{vmatrix} k_1 + k_2 - \omega^2 3m & -k_2 \\ -k_2 & k_2 - \omega^2 m \end{vmatrix} = 0$$

$$3m^2\omega^4 - mk_1\omega^2 - 4mk_2\omega^2 + k_1k_2 = 0$$

substituting k_1, k_2, and m:

$$0.0260419467\ \omega^4 - 32.2963556\ \omega^2 + 3021.95071 = 0$$

Solving this quadratic equation for ω^2 yields

$$\omega_1^2 = \frac{+32.2963556 - \sqrt{1043.05459 - 314.789917}}{0.0520838934}$$

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$$\omega_2^2 = \frac{+32.2963556 + \sqrt{1043.05459 - 314.789917}}{0.0520838934}$$

or,

$$\omega_1^2 = 101.95\ \frac{rad^2}{s^2}$$

$$\omega_2^2 = 1138\ \frac{rad^2}{s^2}$$

$$\omega_1 = 10.097 \frac{rad}{s}$$

$$\omega_2 = 33.734 \frac{rad}{s}$$

$$f_1 = \frac{\omega_1}{2\pi} = 1.607 Hz \qquad T_1 = \frac{1}{f_1} = 0.6223\ Sec$$

$$f_2 = \frac{\omega_2}{2\pi} = 5.369 Hz \qquad T_2 = \frac{1}{f_2} = 0.1863\ Sec$$

Determining 1st Mode Shape:

Plug k_1, k_2, m, and ω_1 = 10.097 rad/sec into $\begin{bmatrix} k_1 + k_2 - \omega_1^2 3m & -k_2 \\ -k_2 & k_2 - \omega_1^2 m \end{bmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\begin{bmatrix} 39.335 + 76.826 - (10.097^2)(3 \cdot 0.09317) & -76.826 \\ -76.826 & 76.826 - (10.097^2)(0.09317) \end{bmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 87.665 & -76.826 \\ -76.826 & 67.327 \end{bmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$87.665\phi_{11} - 76.826\phi_{21} = 0$$

let $\phi_{21} = 1$, then

$$87.665\phi_{11} - 76.826(1) = 0$$

$$\phi_{11} = \frac{76.826}{87.665} = 0.876$$

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$$\Phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0.876 \\ 1.000 \end{Bmatrix}$$

Determining 2nd Mode Shape:

Plug k_1, k_2, m, and ω_2 = 33.734 rad/sec into $\begin{bmatrix} k_1 + k_2 - \omega_2^2 3m & -k_2 \\ -k_2 & k_2 - \omega_2^2 m \end{bmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\begin{bmatrix} 39.335 + 76.826 - (33.734^2)(3 \cdot 0.09317) & -76.826 \\ -76.826 & 76.826 - (33.734^2)(0.09317) \end{bmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -201.917 & -76.826 \\ -76.826 & -29.200 \end{bmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-201.917\phi_{12} - 76.826\phi_{22} = 0$$

let $\phi_{22} = 1$, then

$$-201.917\phi_{12} - 76.826(1) = 0$$

$$\phi_{12} = \frac{76.826}{-201.917} = -0.380$$

$$\Phi_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} -0.380 \\ 1.000 \end{Bmatrix}$$

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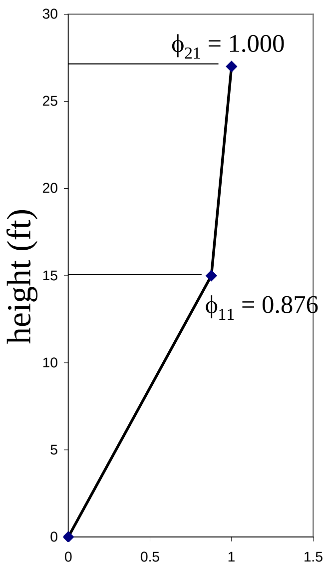
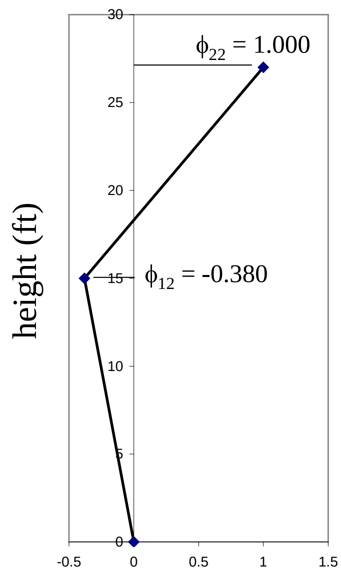


1st Mode Shape



2nd Mode Shape

(b) Verify that the modes are orthogonal as expected.

To verify that the modes are orthogonal, need to show that $\phi_i^T \mathbf{m} \phi_j = \phi_i^T \mathbf{k} \phi_j = 0$, for i≠j

$$\phi_1^T \mathbf{m} \phi_2 = \begin{bmatrix} 0.876 & 1 \end{bmatrix} \begin{bmatrix} (3)0.09317 & 0 \\ 0 & 0.09317 \end{bmatrix} \begin{Bmatrix} -0.380 \\ 1 \end{Bmatrix} = \begin{bmatrix} 0.024485 & 9.317x10^{-2} \end{bmatrix} \begin{Bmatrix} -0.380 \\ 1 \end{Bmatrix} = 0$$

$$\phi_1^T \mathbf{k} \phi_2 = \begin{bmatrix} 0.876 & 1 \end{bmatrix} \begin{bmatrix} 39.335 + 76.826 & -76.826 \\ -76.826 & 76.826 \end{bmatrix} \begin{Bmatrix} -0.380 \\ 1 \end{Bmatrix} = \begin{bmatrix} 24.9310 & 9.5264 \end{bmatrix} \begin{Bmatrix} -0.380 \\ 1 \end{Bmatrix} = 0$$

Can also show $\phi_2^T \mathbf{m} \phi_1 = \phi_2^T \mathbf{k} \phi_1 = 0$

(c) Normalize the first mode such that $\boldsymbol{\phi}_1^{\mathbf{T}} \mathbf{m} \boldsymbol{\phi}_1 = 1.0$

$$\phi_1^T \mathbf{m} \phi_1 = \begin{bmatrix} 0.876 & 1 \end{bmatrix} \begin{bmatrix} (3)0.09317 & 0 \\ 0 & 0.09317 \end{bmatrix} \begin{Bmatrix} 0.876 \\ 1 \end{Bmatrix} = 0.30766$$

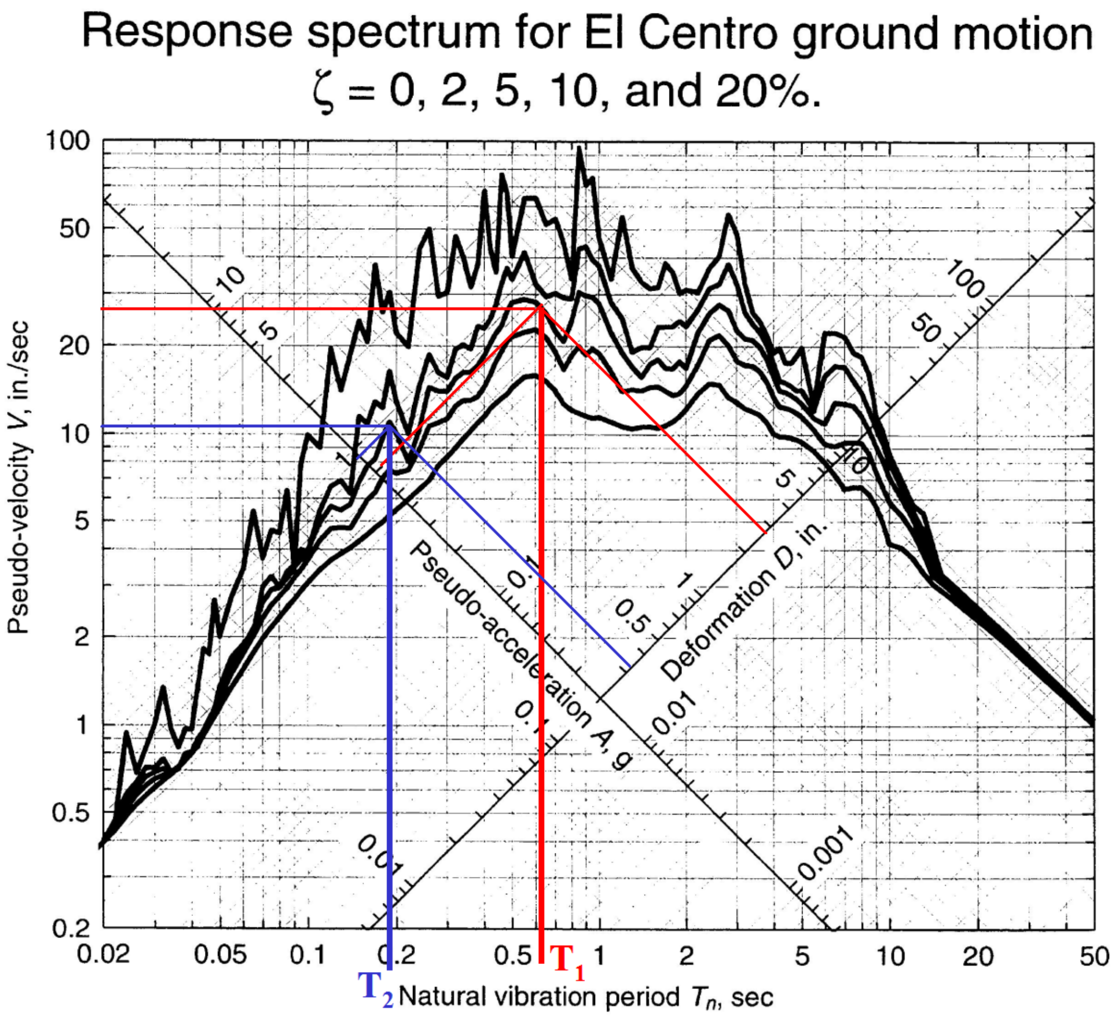
Divide $\boldsymbol{\phi}_1$ by $\sqrt{0.30766}$, therefore $\phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1.5793 \\ 1.8029 \end{Bmatrix}$

Check: $\phi_1^T \mathbf{m} \phi_1 = \begin{bmatrix} 1.5793 & 1.8029 \end{bmatrix} \begin{bmatrix} (3)0.09317 & 0 \\ 0 & 0.09317 \end{bmatrix} \begin{Bmatrix} 1.5793 \\ 1.8029 \end{Bmatrix} = 1.00$ ***ok***

(d) Use the normalized first mode (from above) to verify that $\boldsymbol{\phi}_1^{\mathbf{T}} \mathbf{k} \boldsymbol{\phi}_1 = \omega_1^2$

$$\phi_1^T \mathbf{k} \phi_2 = \begin{bmatrix} 1.5793 & 1.8029 \end{bmatrix} \begin{bmatrix} 39.335 + 76.826 & -76.826 \\ -76.826 & 76.826 \end{bmatrix} \begin{Bmatrix} 1.5793 \\ 1.8029 \end{Bmatrix} = 101.950 \frac{rad^2}{s^2} = \left(10.097 \frac{rad}{s} \right)^2 = \omega_1^2$$

(e) Use the El Centro Response Spectrum and a damping ratio of 5% to estimate the maximum base shear and moment.



From Response Spectrum:

For $T_1 = 0.622$ sec and $\zeta = 5\%$, $D_1 = 2.8$ in. and $A_1 = 0.8g = 309.12$ in/s^2

For $T_2 = 0.0.186$ sec and $\zeta = 5\%$, $D_2 = 0.31$ in. and $A_2 = 0.9g = 347.76$ in/s^2

$$f_{jn} = \frac{L_n}{M_n} A_n m_j \phi_{jn}$$

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$$\frac{L_i}{M_i} = \frac{\sum_{j=1}^{2} m_j \phi_{ji}}{\sum_{j=1}^{2} m_j \phi_{ji}^2}$$

$$\frac{L_1}{M_1} = \frac{\sum_{j=1}^{2} m_j \phi_{j1}}{\sum_{j=1}^{2} m_j \phi_{j1}^2} = \frac{m_1\phi_{11} + m_2\phi_{21}}{m_1\phi_{11}^2 + m_2\phi_{21}^2} = \frac{(0.27951)(0.876) + (0.09317)(1)}{(0.27951)(0.876)^2 + (0.09317)(1)^2} = 1.0987$$

$$\frac{L_2}{M_2} = \frac{\sum_{j=1}^{2} m_j \phi_{j2}}{\sum_{j=1}^{2} m_j \phi_{j2}^2} = \frac{m_1\phi_{12} + m_2\phi_{22}}{m_1\phi_{12}^2 + m_2\phi_{22}^2} = \frac{(0.27951)(-0.380) + (0.09317)(1)}{(0.27951)(-0.380)^2 + (0.09317)(1)^2} = -0.0977$$

$$f_{11} = \frac{L_1}{M_1} A_1 m_1 \phi_{11} = (1.0987)\left(309.12 \frac{in}{s^2}\right)\left(0.27951 \frac{kip \cdot s^2}{in}\right)(0.876) = 83.159 \; kips$$

$$f_{12} = \frac{L_2}{M_2} A_2 m_1 \phi_{12} = (-0.0977)\left(347.76 \frac{in}{s^2}\right)\left(0.27951 \frac{kip \cdot s^2}{in}\right)(-0.380) = 3.609 \; kips$$

$$f_{21} = \frac{L_1}{M_1} A_1 m_2 \phi_{21} = (1.0987)\left(309.12 \frac{in}{s^2}\right)\left(0.09317 \frac{kip \cdot s^2}{in}\right)(1) = 31.643 \; kips$$

$$f_{22} = \frac{L_2}{M_2} A_2 m_2 \phi_{22} = (-0.0977)\left(347.76 \frac{in}{s^2}\right)\left(0.09317 \frac{kip \cdot s^2}{in}\right)(1) = -3.166 \; kips$$

Base Shear

$$V_{0n} = \sum_{j=1}^{N} f_{jn}$$

$$V_{01} = \sum_{j=1}^{2} f_{j1} = f_{11} + f_{21} = 83.159 \; kips + 31.643 \; kips = 114.80 \; kips$$

$$V_{02} = \sum_{j=1}^{2} f_{j2} = f_{12} + f_{22} = 3.609 \; kips - 3.166 \; kips = 0.443 \; kips$$

Calculate Maximum Base Shear, $V_{0\,max}$

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$$V_{0\max} = \sqrt{(V_{01})^2 + (V_{02})^2} = \sqrt{(114.80\ kips)^2 + (0.443\ kips)^2} = 114.80\ kips$$

Base Moment

$$M_{0n} = \sum_{j=1}^{N} f_{jn} d_j$$

$$M_{01} = \sum_{j=1}^{2} f_{j1} d_j = f_{11}d_1 + f_{21}d_2 = 83.159\ kips(180\ in) + 31.643\ kips(324\ in) = 25221.0\ kips \cdot in$$

$$M_{02} = \sum_{j=1}^{2} f_{j2} d_j = f_{12}d_1 + f_{22}d_2 = 3.609\ kips(180\ in) - 3.166\ kips(324\ in) = -376.16\ kips \cdot in$$

Calculate Maximum Base Moment, $M_{0\ max}$

$$M_{0\max} = \sqrt{(M_{01})^2 + (M_{02})^2} = \sqrt{(25221.0\ kips \cdot in)^2 + (-376.16\ kips \cdot in)^2} = 25223.8\ kips \cdot in = 2102.0\ kips \cdot ft$$

(f) Find a_0 and a_1 in $\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k}$ for a viscous damping of 5% in modes 1 and 2.

$\zeta_1 = \zeta_2 = \zeta = 0.05$

$$a_0 = \xi \frac{2\omega_i\omega_j}{\omega_i + \omega_j} = (0.05)\frac{2(10.097\ rad/s)(33.734\ rad/s)}{10.097\ rad/s + 33.734\ rad/s} = 0.777\ rad/s$$

$$a_1 = \xi \frac{2}{\omega_i + \omega_j} = (0.05)\frac{2}{10.097\ rad/s + 33.734\ rad/s} = 0.00228\ \frac{1}{rad/s}$$

$a_0 = 0.777\ rad/s$ & $a_1 = 0.00228\ s/rad$