

Single-Degree-of-Freedom (SDOF) and Response Spectrum

Ahmed Elgamal

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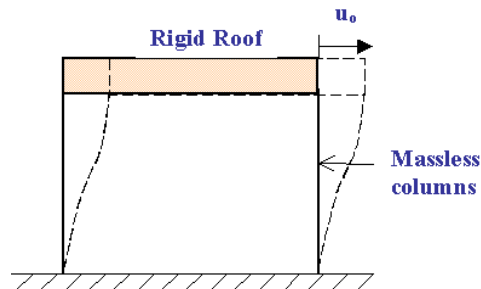
Dynamics of a Simple Structure

The Single-Degree-Of-Freedom (SDOF) Equation

References

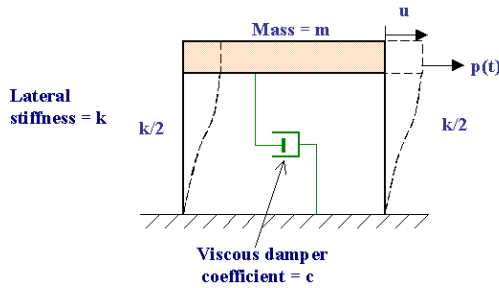
Dynamics of Structures, Anil K. Chopra, Prentice Hall, New Jersey, ISBN 0-13-855214-2 (Chapter 3).

Elements of Earthquake Engineering And Structural Dynamics, André Filiatrault, Polytechnic International Press, Montréal, Canada, ISBN 2-553-00629-4 (Section 4.2.3).

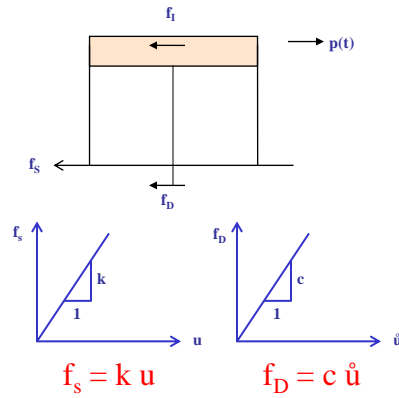


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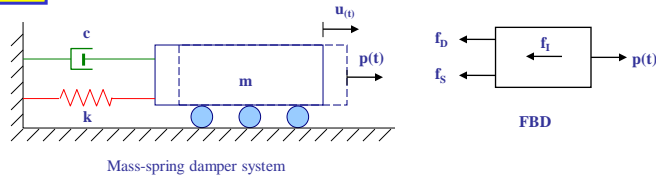
Equation of motion (external force)



Free-Body Diagram (FBD)



$m\ddot{u} + c\dot{u} + ku = p(t)$



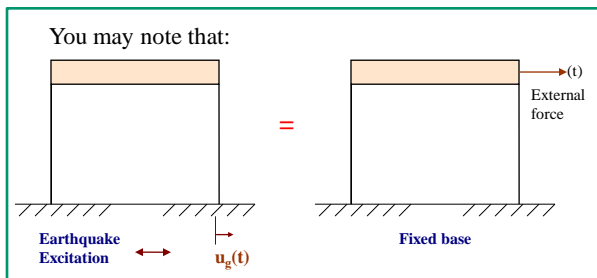
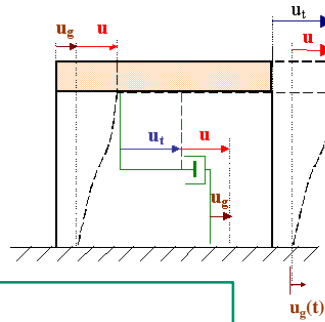
Earthquake Ground Motion (u_g)

$f_I + f_D + f_S = 0$

$f_I = m\ddot{u}_t = m(\ddot{u}_g + \ddot{u})$

$m(\ddot{u} + \ddot{u}_g) + c\dot{u} + ku = 0$

$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$



u = relative displacement (displacement of the structure relative to the ground)
 u_t = total displacement
 $f_I = m \times$ (total or absolute acceleration)

Undamped natural frequency

Property of structure when allowed to vibrate freely without external excitation

$$\omega = \sqrt{\frac{k}{m}} \quad \text{Undamped natural circular frequency of vibration (radians/second)}$$

$$f = \frac{\omega}{2\pi} \quad \text{natural cyclic frequency of vibration (cycles/second or 1/second or Hz)}$$

$$T = \frac{1}{f} \quad \text{natural period of vibration (second)}$$

T is the time required for one cycle of free vibration

If damping is present, replace ω by ω_D

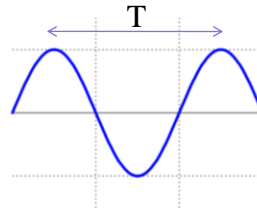
where $\omega_D = \omega\sqrt{1 - \zeta^2}$ natural frequency*, and

$$\zeta = \frac{c}{2m\omega} \quad \text{fraction of critical damping coefficient (**damping ratio**, zeta)}$$

$$= \frac{c}{c_c} = \frac{c}{2\sqrt{km}} \quad \text{(dimensionless measure of damping)}$$

$$c_c = 2m\omega = 2\sqrt{km} \quad c_c = \text{critical damping}$$

* Note: In earthquake engineering, $\omega_D = \omega$ approximately, since ζ is usually below 0.2 (or 20%)

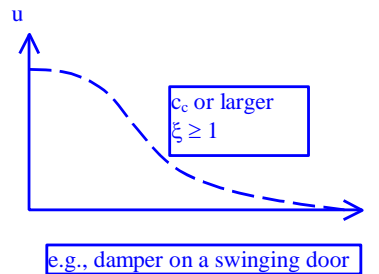


c_c is the least level of damping that prevents oscillation

In general $\zeta < 0.2$, i.e., $\omega_D \approx \omega$, $f_D \approx f$, $T = T_D$

ζ may be in the range of 0.02 – 0.2 or 2% - 20%

5% is sometimes a typical value.



in terms of ζ

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$$

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = -\ddot{u}_g$$

$$\ddot{u} + 2\zeta\omega\dot{u} + \omega^2u = -\ddot{u}_g$$

Note: After the phase of forced vibration (due to external force or base excitation, or initial conditions), the structure continues to vibrate in a “free vibration” mode till it stops due to damping. The ratio between amplitude in two successive cycles is

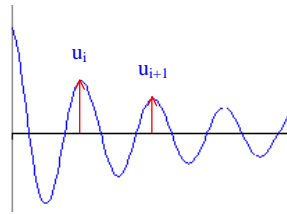
$$\frac{u_i}{u_{i+1}} \approx e^{2\pi\xi}$$

where we define the logarithmic decrement as

$$\delta = 2\pi\xi = \ln\left(\frac{u_i}{u_{i+1}}\right) \text{ if you measure a free vibration response you can find } \xi.$$

Note: for peaks j cycles apart

$$\ln\left(\frac{u_i}{u_{i+j}}\right) = j\delta = 2j\pi\xi$$



Free vibration

Why is $c_c = 2m\omega = 2\sqrt{km}$

Critical viscous damping

The free vibration equation may be written as

$$m\ddot{x} + c\dot{x} + kx = 0$$

and the general solution is

$$x = C_1 e^{\left[\frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\right] t} + C_2 e^{\left[\frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\right] t}$$

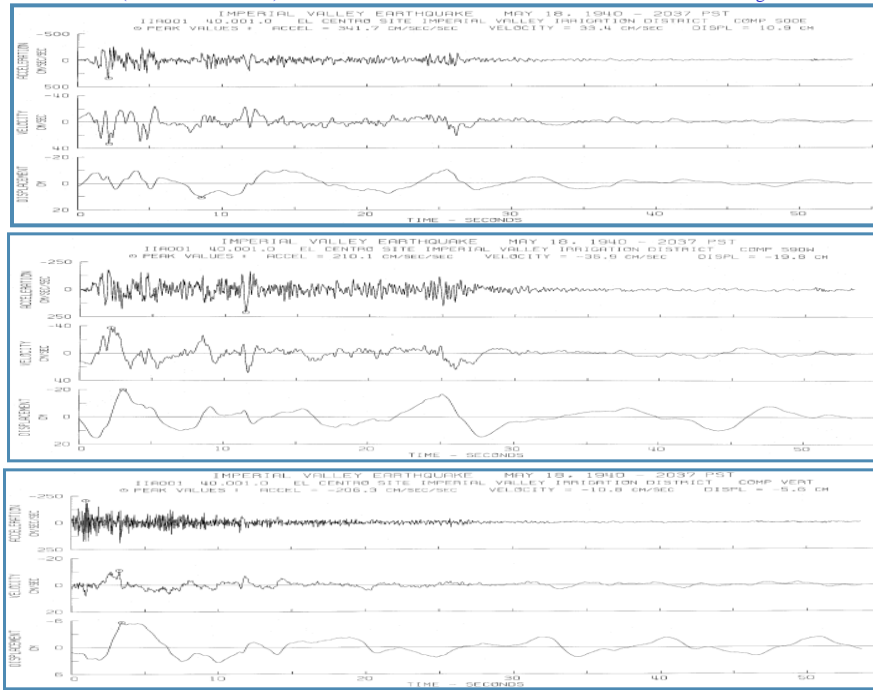
if $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$, the radical part of the exponent will vanish. This will produce aperiodic

response (non-oscillatory). In this case

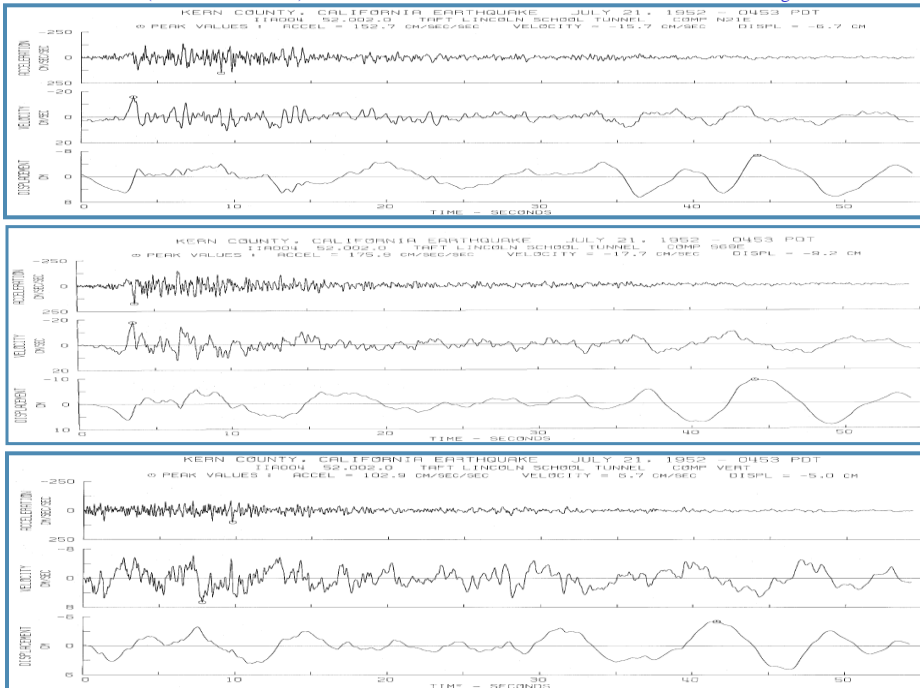
$$\frac{c^2}{(2m)^2} = \frac{k}{m} \text{ or } c = 2\sqrt{km} = c_c$$

since $\omega = \sqrt{\frac{k}{m}}$, c_c is also equal to $2m\omega$ (note that $2\sqrt{km} = 2\sqrt{m\omega^2 m} = 2m\omega$)

$$\text{and also } c_c = 2\sqrt{km} = 2\sqrt{k(k/\omega^2)} = \frac{2k}{\omega}$$



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Deformation Response Spectrum

For a given EQ excitation calculate $|u_{\max}|$ from SDOF response with a certain ξ and within a range of natural periods or frequencies.

$|u_{\max}|$ for each frequency will be found from the computed $u(t)$ history at this frequency.

A plot of $|u_{\max}|$ vs. natural period is constructed representing the deformation (or displacement) response spectrum (S_d).

From this figure, one can directly read the maximum relative displacement of any structure of natural period T (and a particular value of ξ as damping)

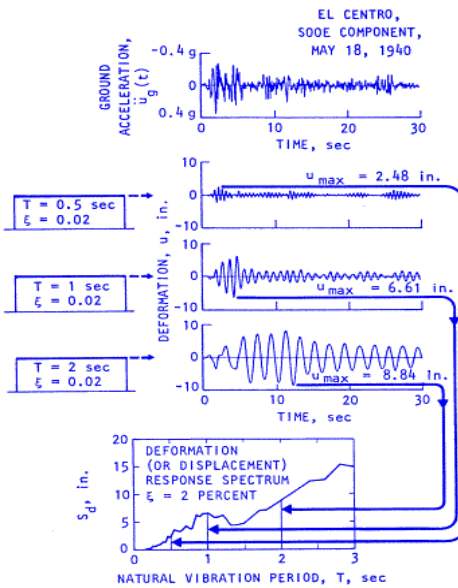


Figure 17. Computation of deformation (or displacement) response spectrum

From: Chopra, Dynamics of Structures, A Primer

Example

Units for natural frequency calculation (SI units)

Weight = 9.81 kN = 9810 N = 9810 kg (m/s²)

Gravity (g) = 9.81m/s²

Mass (m) = W/g = 9180/9.81 = 1000 kg

Stiffness (k) = 81 kN/m = 81000 N/m = 81000 kg (m/s²)/m = kg /s²

ω = SQRT (k/m) = SQRT (81000/1000) = 9 radians/sec

f = $\omega / (2 \pi)$ = 1.43 Hz (units of 1/s or cycles/sec)

Units for natural frequency calculation (English units)

Weight (W) = 193.2 Tons = 193.2 (2000) = 386,400 lbs = 386.4 kips

g = 386.4 in/sec/sec (or in/s²)

Mass (m) = W/g = 1.0 kips s²/in

Stiffness (k) = 144 kips/in

ω = SQRT (k/m) = 12 radians/sec

f = $\omega / (2 \pi)$ = 1.91 Hz (units of 1/s or cycles/sec)

Note: The weight of an object is the force of gravity on the object and may be defined as the mass times the acceleration of gravity, $w = mg$. Weight is what is measured by a scale (e.g., weight of a person). Since the weight is a force, its SI unit is Newton.

Concept of Equivalent lateral force f_s

If f_s is applied as a static force, it would cause the deformation u .

Thus at any instant of time:

$$f_s = ku(t), \text{ or in terms of the mass } f_s(t) = m\omega^2u(t)$$

$$\omega = \sqrt{\frac{k}{m}}$$

The maximum force will be

$$f_{s,max} = m\omega^2u_{max} = ku_{max} = mS_a = kS_d$$

$$S_a = \frac{k}{m}S_d \quad S_a = \omega^2S_d$$

S_d = deformation or displacement response spectrum

$$S_a = \omega^2S_d = \text{pseudo-acceleration response spectrum}$$

The maximum strain energy E_{max} stored in the structure during shaking is:

$$E_{max} = \frac{1}{2}ku_{max}^2 = \frac{1}{2}kS_d^2 = \frac{1}{2}\frac{k}{\omega^2}\omega^2S_d^2 = \frac{1}{2}m\omega^2S_d^2 = \frac{1}{2}mS_v^2$$

where $S_v = \omega S_d$ = pseudo-velocity response spectrum

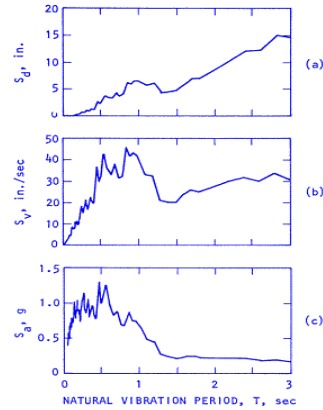


Figure 19. (a) Deformation (or Displacement), (b) pseudo-velocity and (c) pseudoacceleration response spectra. El Centro ground motion—S00°E component. Damping ratio $\xi = 2$ percent

From: Chopra, Dynamics of Structures, A Primer

Note that S_d , S_v and S_a are inter-related by

$$S_a = \omega^2S_d = \omega S_v$$

S_a , S_v are directly related to S_d

by ω^2 and ω respectively or by $(2\pi f)^2$ and $2\pi f$;

or $\left(\frac{2\pi}{T}\right)^2$ and $\left(\frac{2\pi}{T}\right)$ as shown in Figure.

Due to this direct relation, a 4-way plot is usually used to display S_a , S_v and S_d on a single graph as shown in Figure.

In this figure, the logarithm of period T , S_a , S_v and S_d is used.

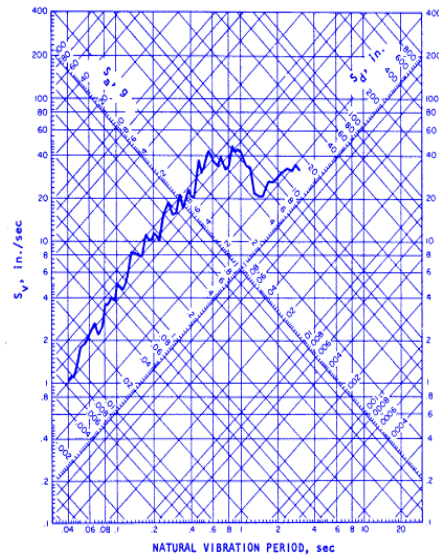


Figure 20. Four-way logarithmic plot of response spectrum. El Centro ground motion—S00°E component. Damping ratio $\xi = 2$ percent

From: Chopra, Dynamics of Structures, A Primer

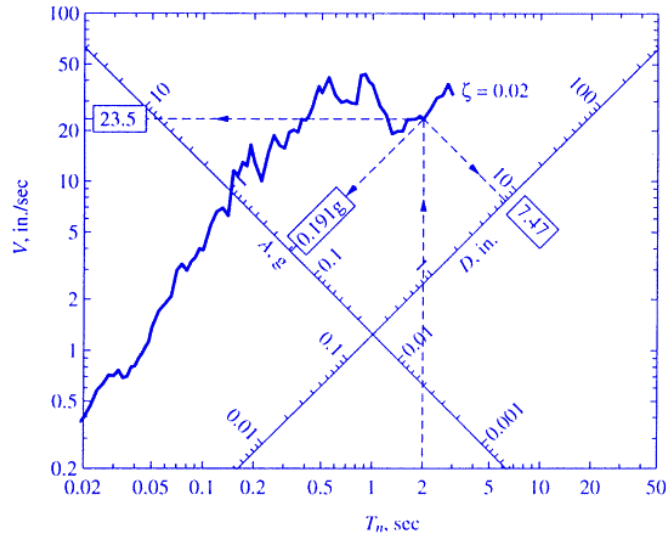


Figure 6.6.3 Combined *D-V-A* response spectrum for El Centro ground motion; $\zeta = 2\%$.

From: Chopra, Dynamics of Structures

In order to cover the damping

range of interest,

it is common to perform

the same calculations for

$\xi = 0.0, 0.02, 0.05, 0.10,$

and 0.20 (see Figure)

Typical Notation:

$$S_v \equiv \text{PSV} \equiv V$$

$$S_a \equiv \text{PSA} \equiv A$$

$$S_d \equiv \text{SD} \equiv D$$

Example (El-Centro motion):
 Find maximum displacement
 and base shear of tower
 with $f = 2 \text{ Hz}$, $\xi = 2\%$
 and $k = 1.5 \text{ MN/m}$

$$\text{Period } T = 1/f = 0.5 \text{ second}$$

$$S_d = 2.5 \text{ inches} = 0.0635 \text{ m}$$

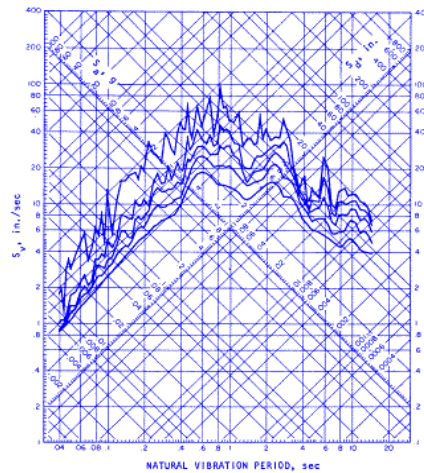
$$\text{Force}_{\max} = k u_{\max}$$

$$= 1.5 \text{ MN/m} \times 0.0635 \text{ m} = 95.25 \text{ kN}$$

RESPONSE SPECTRUM

IMPERIAL VALLEY EARTHQUAKE
 MAY 18, 1940 — 2037 PST

111A001 40.001.0 EL CENTRO SITE
 IMPERIAL VALLEY IRRIGATION DISTRICT COMP 500E
 DAMPING VALUES ARE 0, 2, 5, 10, AND 20 PERCENT OF CRITICAL

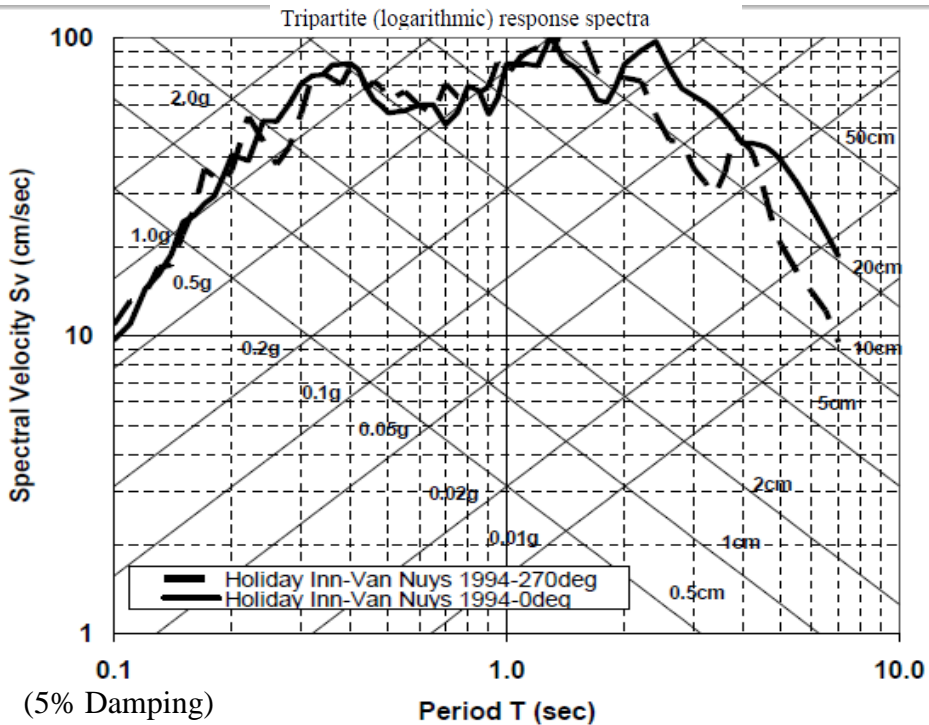
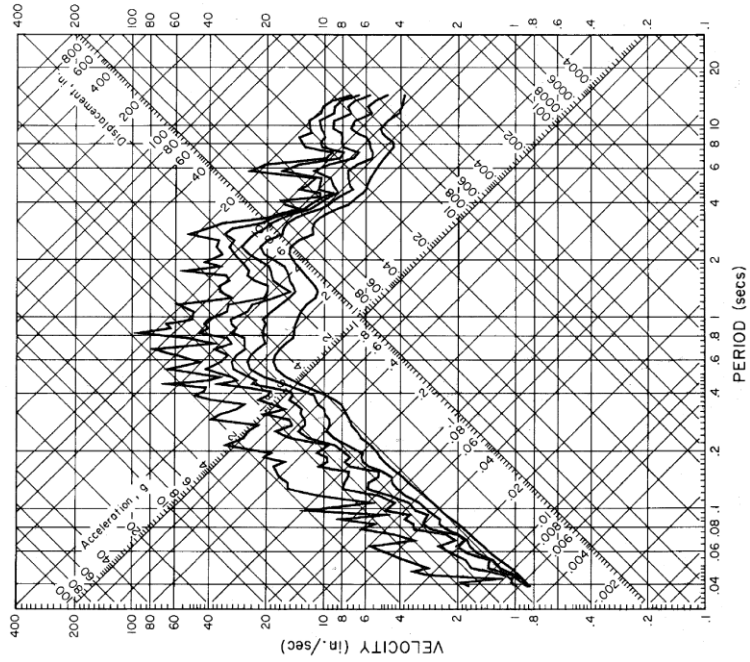


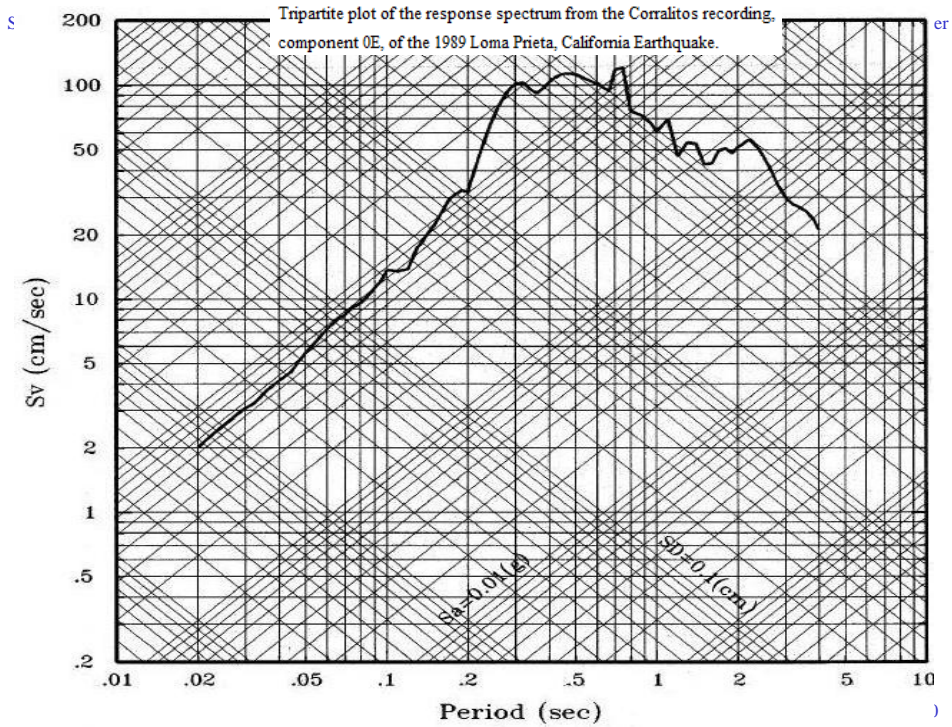
From: Chopra, Dynamics of Structures, A Primer

Figure 21. Four-way logarithmic plot of response spectrum, El Centro ground motion— $S_{00^\circ E}$ component (after Hudson, 1979)

RESPONSE SPECTRUM

IMPERIAL VALLEY EARTHQUAKE MAY 18, 1940 - 2037 PST
 111P001 40.001.0 EL CENTRO SITE IMPERIAL VALLEY IRRIGATION DISTRICT COMP CODE
 DAMPING VALUES ARE 0.2, 5, 10 AND 20 PERCENT OF CRITICAL





First version: September 2003 (Last modified 2010)

Ahmed Elgamal & Mike Fraser

Inspection of this figure shows that the maximum response at short period (high frequency stiff structure) is controlled by the ground acceleration, low frequency (long period) by ground displacement, and intermediate period by ground velocity.

```
Get copy of El-Centro
(May 18, 1940)
earthquake record
S00E (N-S component)
ftp nisee.ce.berkeley.edu
(128.32.43.154)
login: anonymous
password: your_indent
cd pub/a.k.chopra
get el_centro.data
quit
```

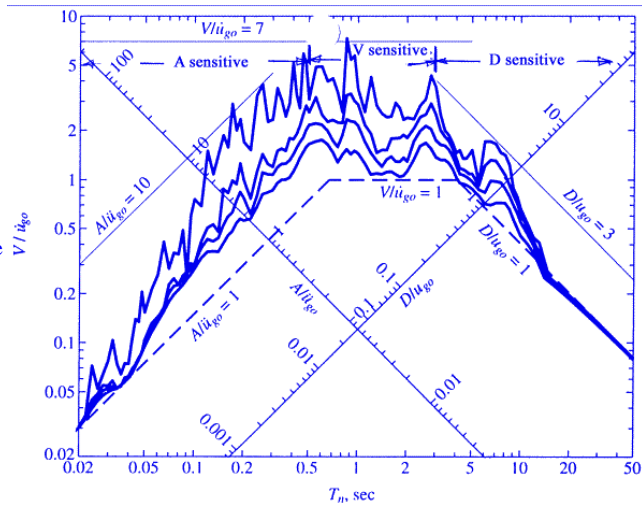


Figure 6.8.2 Response spectrum for El Centro ground motion plotted with normalized scales $A/i\ddot{u}_{go}$, $V/i\dot{u}_{go}$, and D/iu_{go} ; $\zeta = 0, 2, 5,$ and 10% .

From: Chopra, Dynamics of Structures

Note that response spectrum for relative velocity may be obtained from the SDOF response history, and similarly for $\dot{u}^1 = (\dot{u} + \dot{u}_g)$. These spectra are known as relative velocity and total acceleration response spectra, and are different from the pseudo velocity and pseudo acceleration spectra S_v and S_a (which are directly related to S_d).
 e.g. for $\xi = 0\%$
 $m(\ddot{u} + \ddot{u}_g) + k u = 0$
 or $(\ddot{u} + \ddot{u}_g) + \omega^2 u = 0$

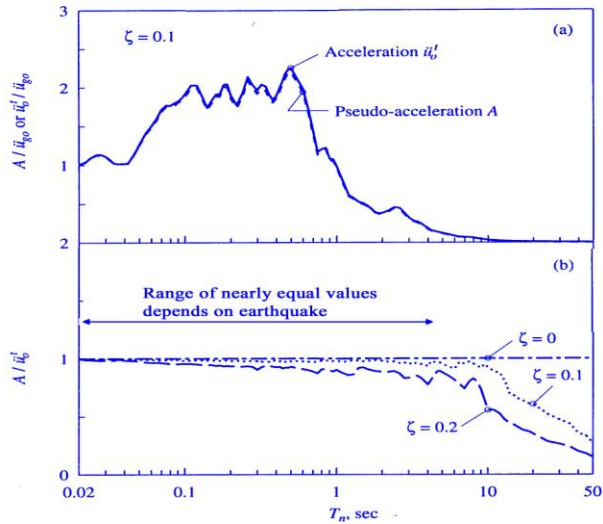
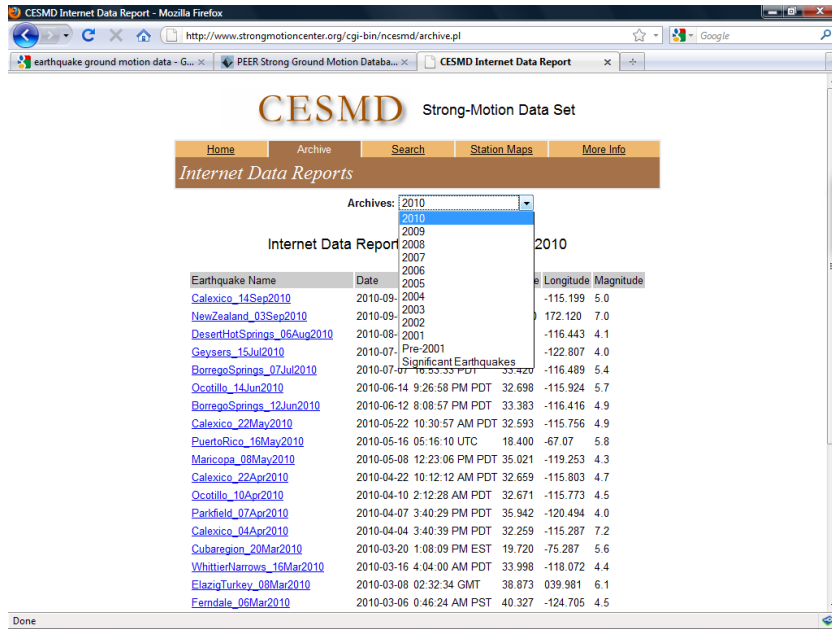
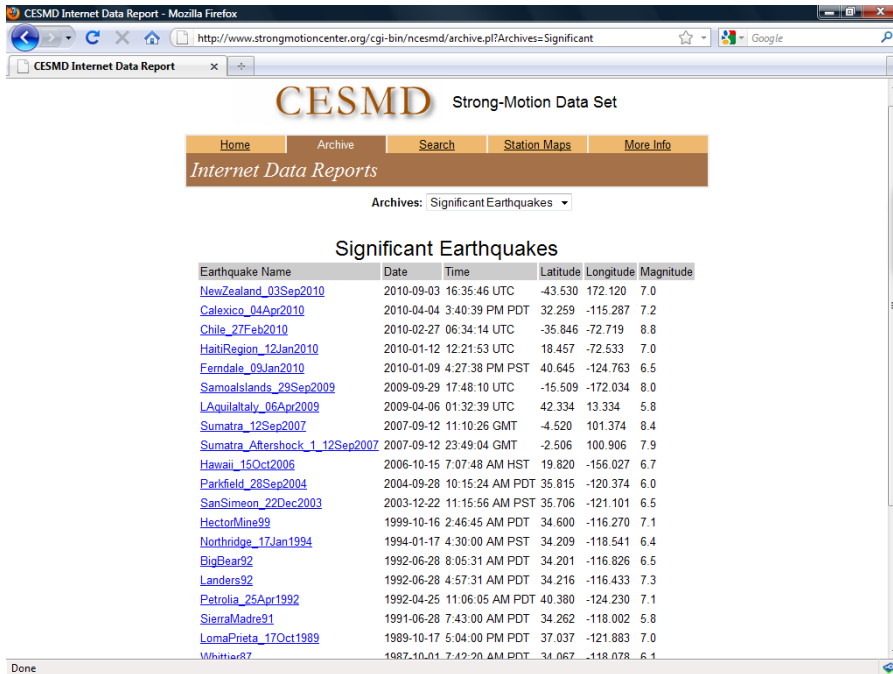


Figure 6.12.2 (a) Comparison between pseudo-acceleration and acceleration response spectra; $\xi = 10\%$; (b) ratio A/u_g'' for $\xi = 0, 10,$ and 20% .

From: Chopra, Dynamics of Structures



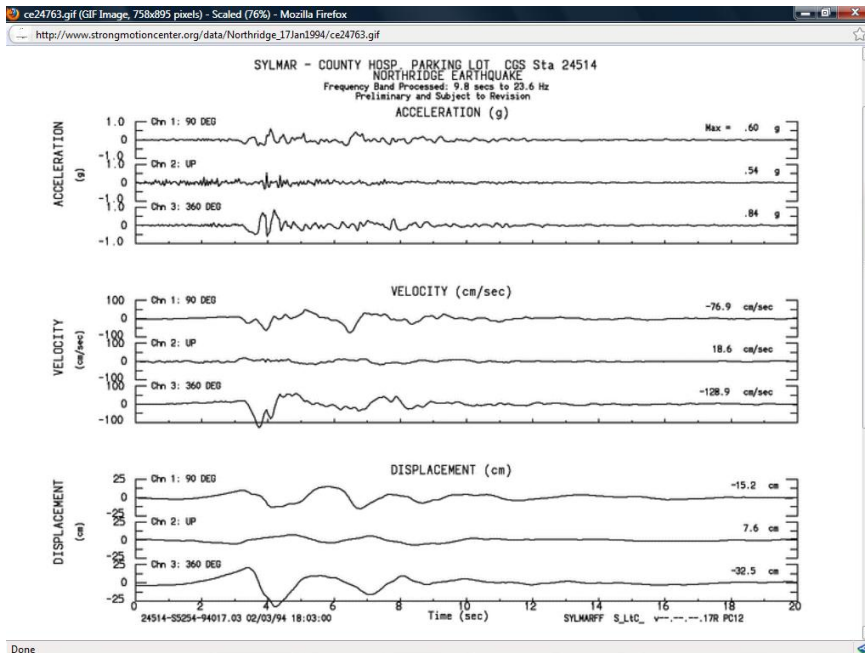
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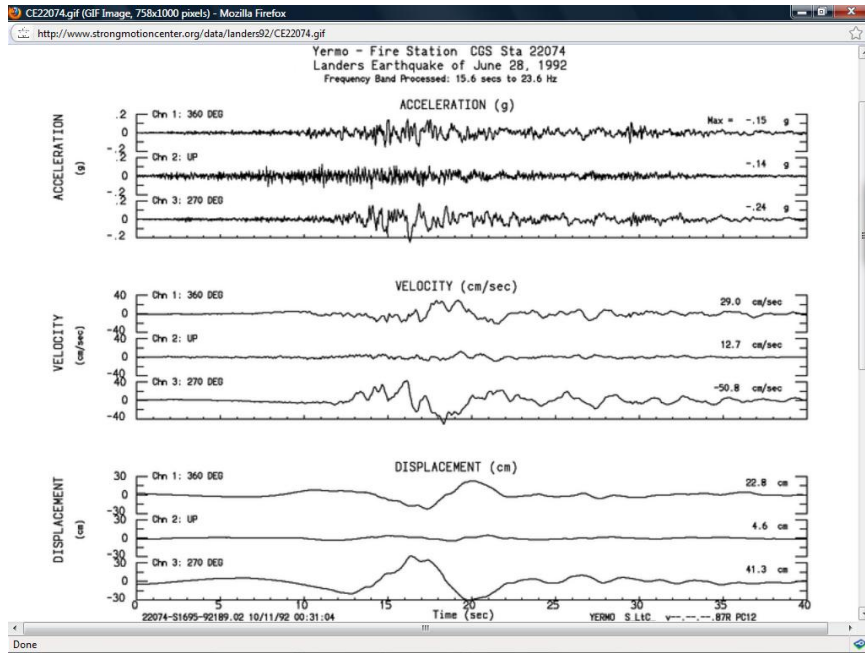
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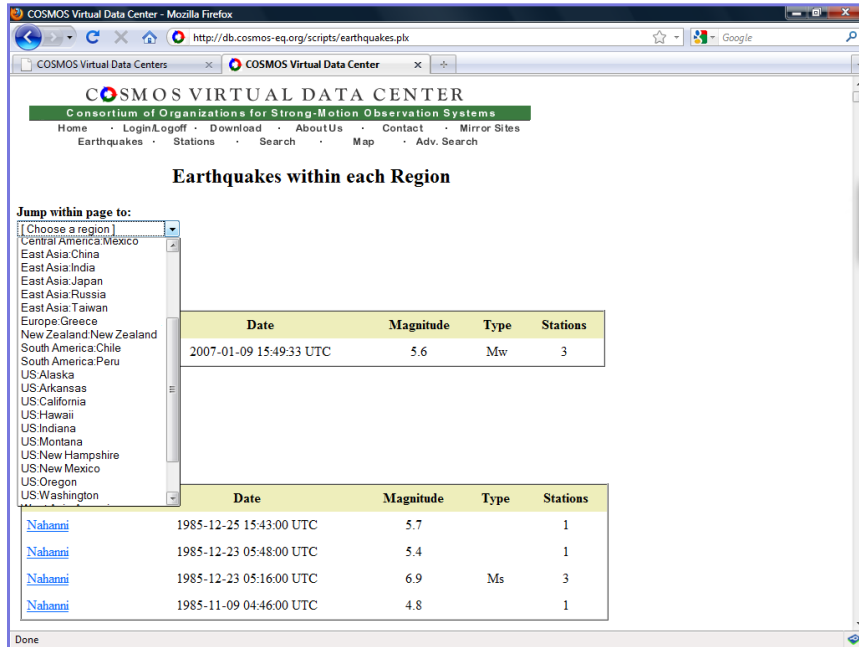
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COSMOS Virtual Data Center - Mozilla Firefox
 http://db.cosmos-eq.org/scripts/earthquakes.pl#CA

California

Earthquake	Date	Magnitude	Type	Stations
Lafayette	2007-03-02 04:40:00 UTC	4.2	Mw	56
Offshore Northern California	2007-02-26 12:19:54 UTC	5.4	Mw	11
Offshore Northern California	2006-07-19 11:41:43 UTC	5.0	Mw	3
Obsidian Butte	2005-09-02 01:27:19 UTC	5.1	Mw	12
Greater Los Angeles	2005-06-16 20:53:26 UTC	4.9	Mw	55
Off the Coast of Northern California	2005-06-15 02:50:54 UTC	7.2	Mw	14
Anza	2005-06-12 15:41:46 UTC	5.2	Mw	121
Mettler	2005-04-16 19:18:13 UTC	4.6	Mw	12
Parkfield Aftershock	2004-09-30 18:54:28 UTC	5.0	Mw	3
Keene	2004-09-29 22:54:54 UTC	5.0	ML	1
Parkfield	2004-09-28 17:15:24 UTC	6.0	Mw	97
Adobe Hills	2004-09-18 23:43:31 UTC	5.4	Mw	8
Adobe Hills	2004-09-18 23:02:17 UTC	5.5	Mw	8
San Clemente Island	2004-06-15 22:28:49 UTC	5.2	ML	23
San Simeon	2003-12-22 19:15:56 UTC	6.5	Mw	71
Humboldt Hill	2003-08-15 09:22:13 UTC	5.1	ML	1
Big Bear City	2003-02-22 12:19:00 UTC	5.4	ML	113
Yreka Lake	2003-02-02 07:08:00 UTC	4.8	ML	22

Done

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COSMOS: Northridge 1994-01-17 12:30:55 UTC - Mozilla Firefox
 http://db.cosmos-eq.org/scripts/event.pl?evnt=21

COSMOS VIRTUAL DATA CENTER
 Consortium of Organizations for Strong-Motion Observation Systems

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 Earthquakes · Stations · Search · Map · Adv. Search

Northridge 1994-01-17 12:30:55 UTC

Region: California
Latitude: 34.2057
Longitude: -118.5539
Depth: 17.50 km
Mechanism: Reverse
Strike: 122
Dip: 40
Rake: 104
Seismic Moment: 1.21618600064638e+26
ML: 6.4
Mw: 6.7
Ms: 6.8
[References](#)

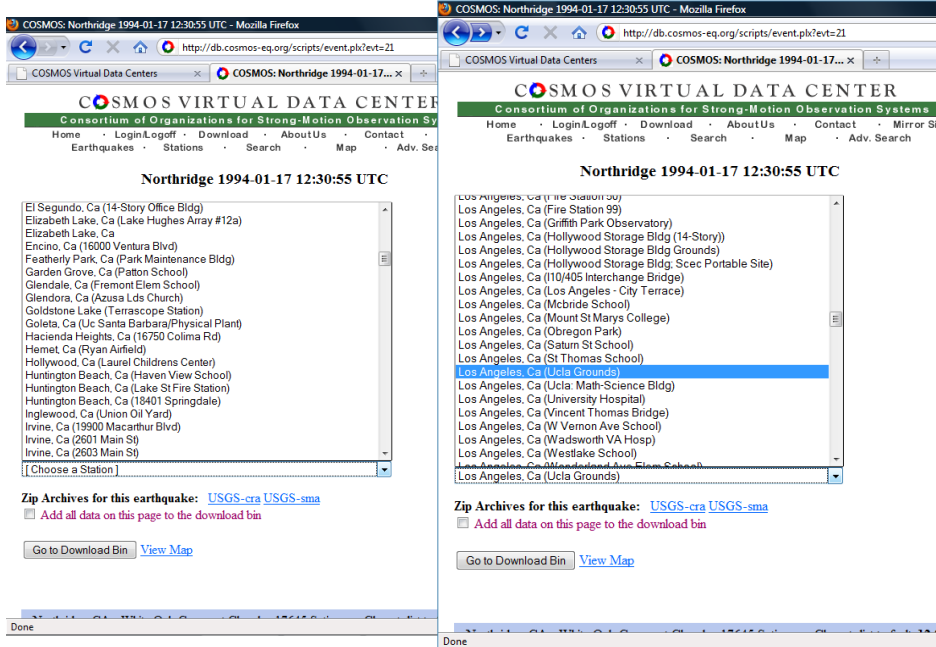
Jump within page to:
 [[Choose a Station]]

Zip Archives for this earthquake: [USGS-cra](#) [USGS-sma](#)
 Add all data on this page to the download bin

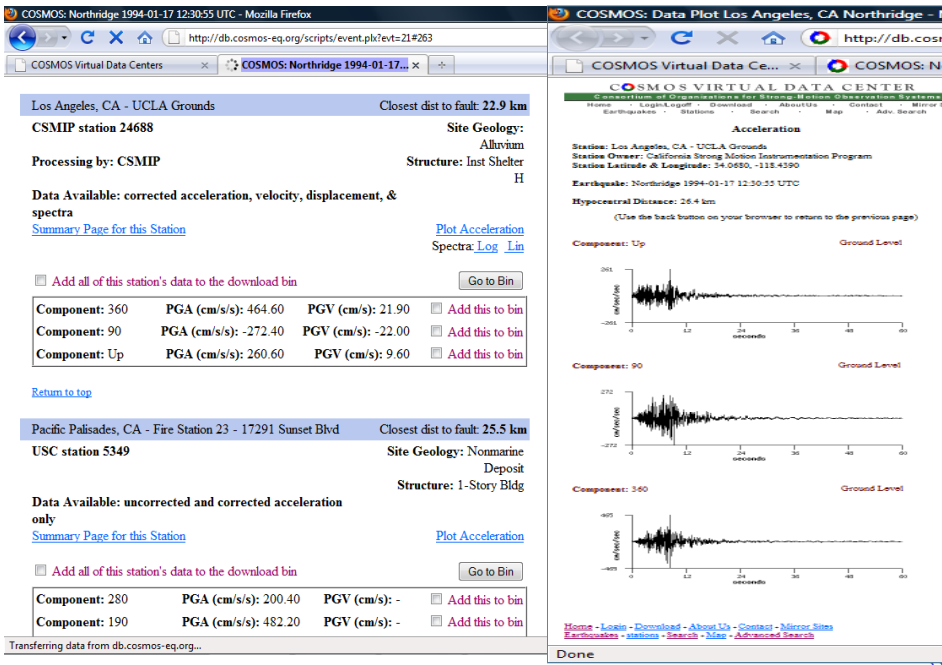
[Go to Download Bin](#) [View Map](#)

Done

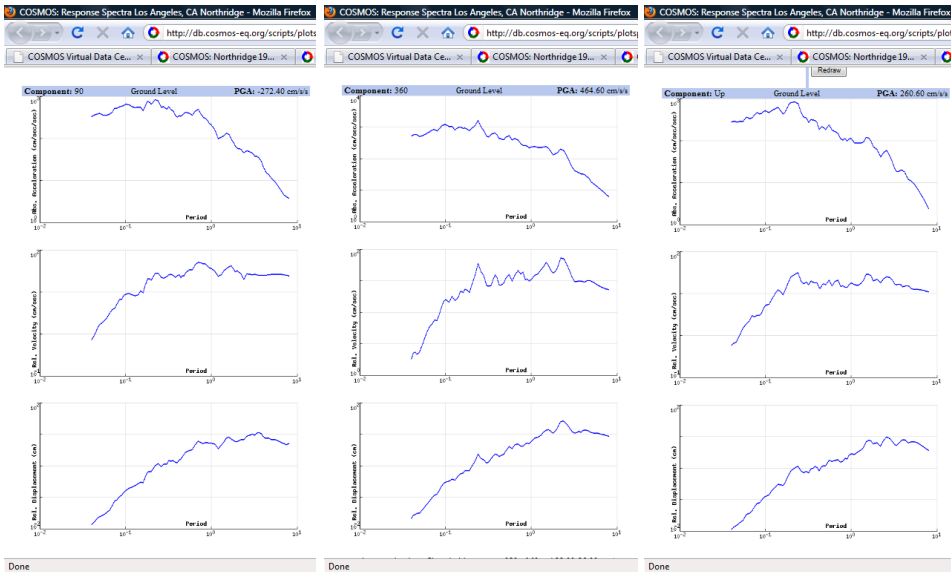
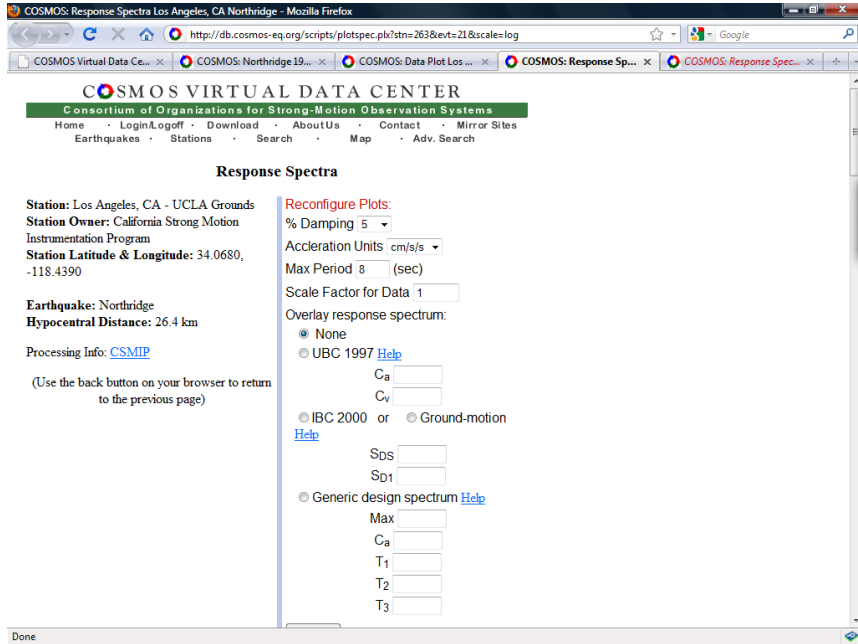
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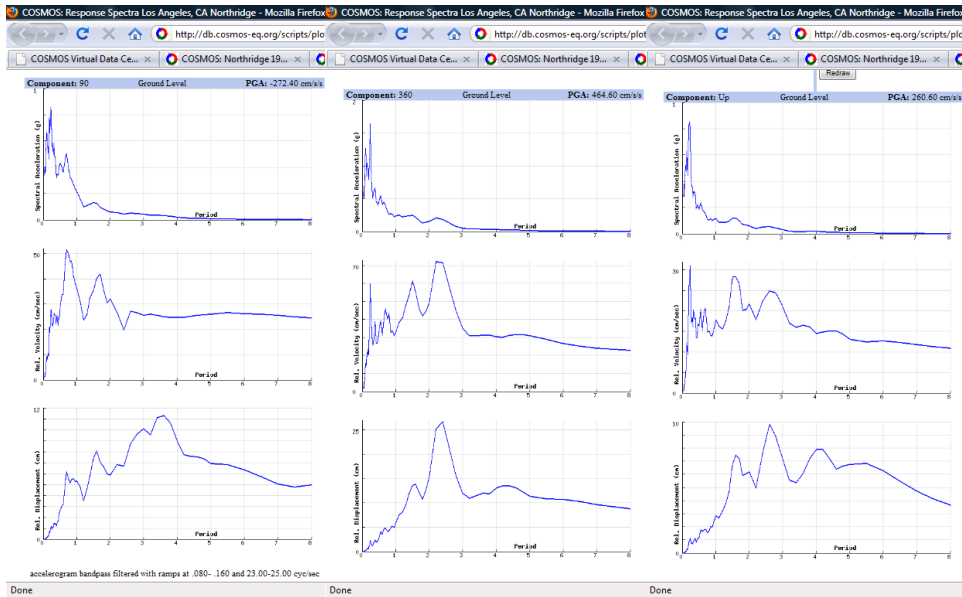


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Elastic Design Spectrum

- Use recorded ground motions (available)
- Use ground motions recorded at similar sites:
 - Magnitude of earthquake
 - Distance of site form earthquake fault
 - Fault mechanism
 - Local Soil Conditions
 - Geology/travel path of seismic waves

Motions recorded at the same location. For design, we need an envelope. One way is to take the average (mean) of these values (use statistics to define curves for mean and standard deviation, see next page)

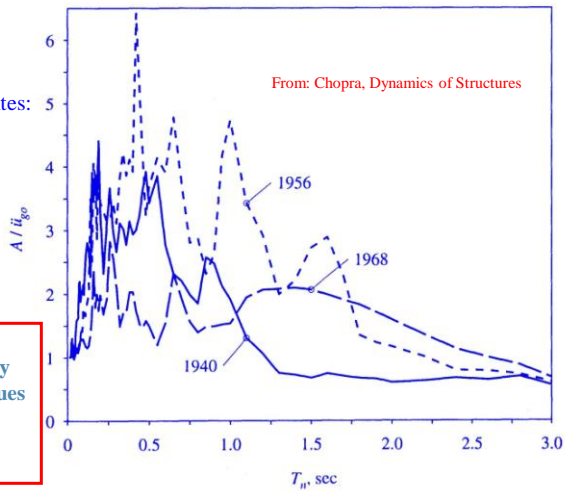
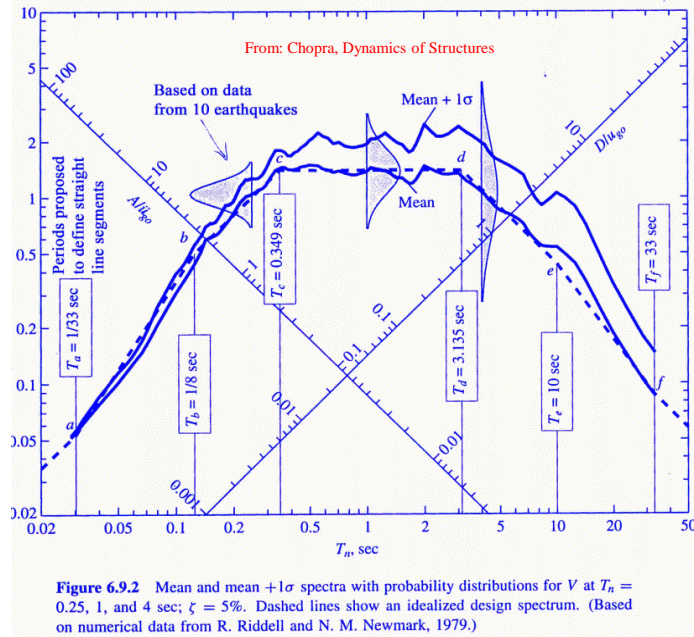
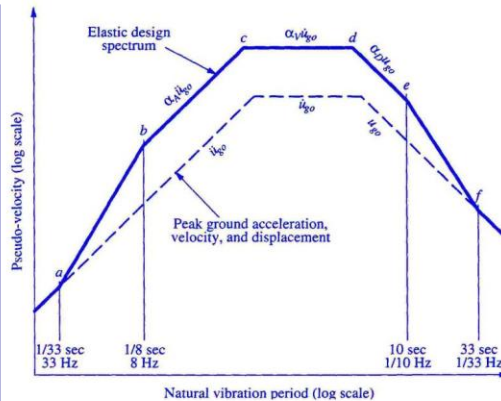


Figure 6.9.1 Response spectra for the north-south component of ground motions recorded at the Imperial Valley Irrigation District substation, El Centro, California, during earthquakes of May 18, 1940; February 9, 1956; and April 8, 1968. $\zeta = 2\%$.



Mean response spectrum is smooth relative to any of the original contributing spectra 38

As an alternative Empirical approach, the periods shown on the this Figure and the values in the Table can be used to construct a Median elastic design spectrum (or a Median + one Sigma), where Sigma is the Standard Deviation. **For any geographic location, these spectra are built based on an estimate of peak ground acceleration, velocity, and displacement as this location.**



From: Chopra, Dynamics of Structures

Figure 6.9.3 Construction of elastic design spectrum.

The periods and values in Table 6.9.1

were selected to give a good match

to a statistical curve such as Figure 6.9.2

based on an ensemble

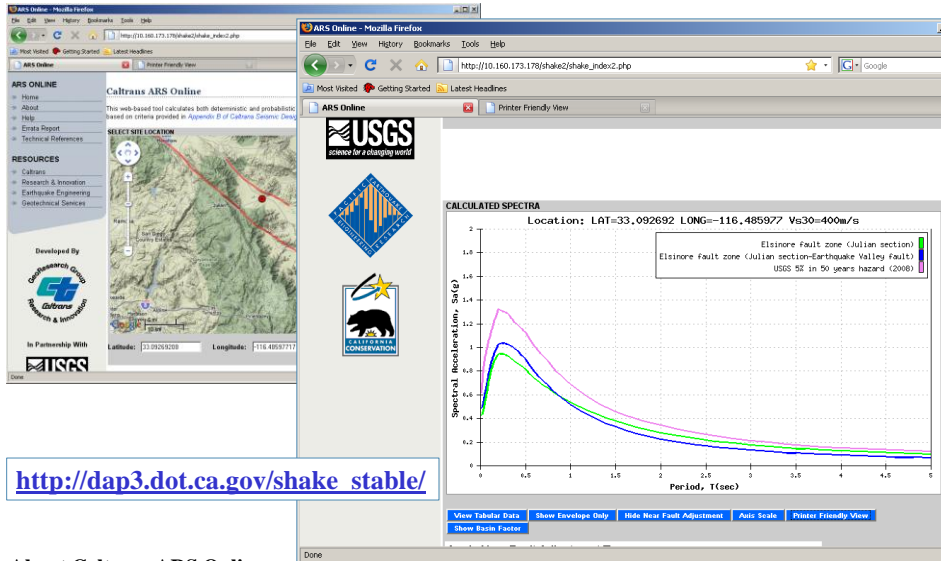
of 50 earthquakes on competent soils.

TABLE 6.9.1 AMPLIFICATION FACTORS: ELASTIC DESIGN SPECTRA

Damping, ζ (%)	Median (50 percentile)			One Sigma (84.1 percentile)		
	α_A	α_V	α_D	α_A	α_V	α_D
1	3.21	2.31	1.82	4.38	3.38	2.73
2	2.74	2.03	1.63	3.66	2.92	2.42
5	2.12	1.65	1.59	2.71	2.30	2.01
10	1.64	1.37	1.20	1.99	1.84	1.69
20	1.17	1.08	1.01	1.26	1.37	1.38

Source: N. M. Newmark and W. J. Hall, *Earthquake Spectra and Design*, Earthquake Engineering Research Institute, Berkeley, Calif., 1982, pp. 35 and 36.

(Design Spectrum may include more than one Earthquake fault scenario)



About Caltrans ARS Online

This web-based tool calculates both deterministic and probabilistic acceleration response spectra for any location in California based on criteria provided in *Appendix B of Caltrans Seismic Design Criteria*