STEP-BY-STEP PROCEDURE FOR SETTING UP A SPREADSHEET FOR USING NEWMARK'S METHOD AND MODAL ANALYSIS TO SOLVE FOR THE RESPONSE OF A MULTI-DEGREE OF FREEDOM (MDOF) SYSTEM

Start with the equation of motion for a linear multi-degree of freedom system with base ground excitation:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\mathbf{1}\ddot{u}_g$$

Using Modal Analysis, we can rewrite the original coupled matrix equation of motion as a set of un-coupled equations.

$$\ddot{q}_i + 2\zeta\omega\dot{q}_i + \omega_i^2 q_i = -\frac{L_i}{M_i}\ddot{u}_g, \ \ i = 1, 2, \ldots, \text{NDOF}$$

with initial conditions of $d_i(t=0) = d_{io}$ and $v_i(t=0) = v_{io}$

Note that total acceleration or absolute acceleration will be $\ddot{q}_{i\,abs} = \ddot{q}_i + \ddot{u}_g$

We can solve each one separately (as a SDOF system), and compute histories of q_i and their time derivatives. To compute the system response, plug the q vector back into $\mathbf{u} = \mathbf{\Phi q}$ and get the u vector (and the same for the time derivatives to get velocity and acceleration).

The beauty here is that there is no matrix operations involved, since the matrix equation of motion has become a set of un-coupled equation, each including only one generalized coordinate q_n.

In the spreadsheet, we will solve each mode in a separate worksheet.

Step 1 - Define System Properties and Initial Conditions for First Mode

(A) Begin by setting up the cells for the Mass, Stiffness, and Damping of the SDOF System (Fig. 1). These values are known.

(B) Set up the cells for the modal participation factor $\frac{L_i}{M_i}$ and mode shape ϕ_i (Fig. 1).

These values must be determined in advance using Modal Analysis.

(C) Calculate the Natural Frequency of the SDOF system using the equation

$$\omega_i = \sqrt{K_i / M_i} \qquad \text{(Equation 1)}$$

Note: If the system damping is given in terms of the Modal Damping Ratio (ζ_i) then the Damping (c) can be calculated using the equation:

$$C_i = 2\, \zeta_i\, \omega_i\, M_i \quad \text{(Equation 2)}$$

(D) Set up the cells for the 2 Newmark Coefficients α & β (Fig. 1), which will allow for performing

a) the Average Acceleration Method, use $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$.

b) the Linear Acceleration Method, use $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{6}$.

(E) Set up cells (Fig. 1) for the initial displacement and velocity (d_o and v_o respectively)

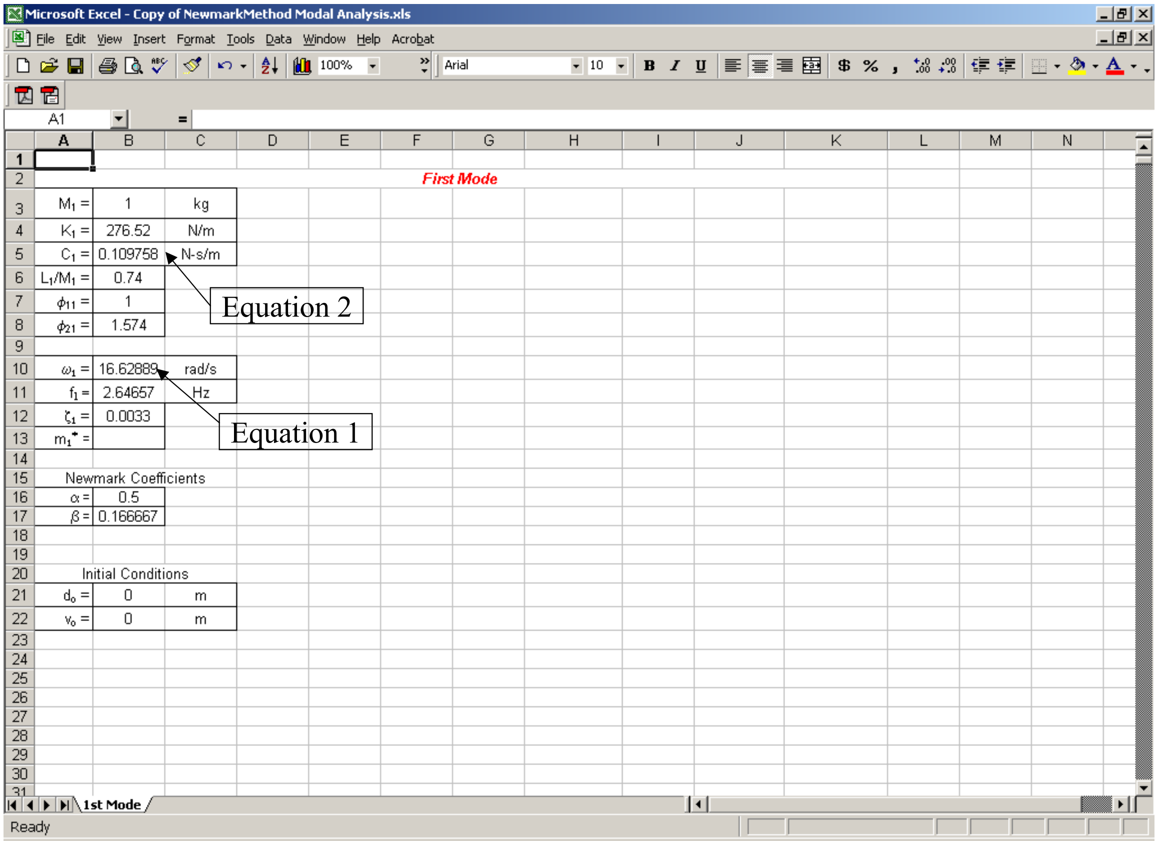


Figure 1: Spreadsheet After Completing Step 1

Step 2 – Set Up Columns for Solving The Equation of Motion Using Newmark's Method

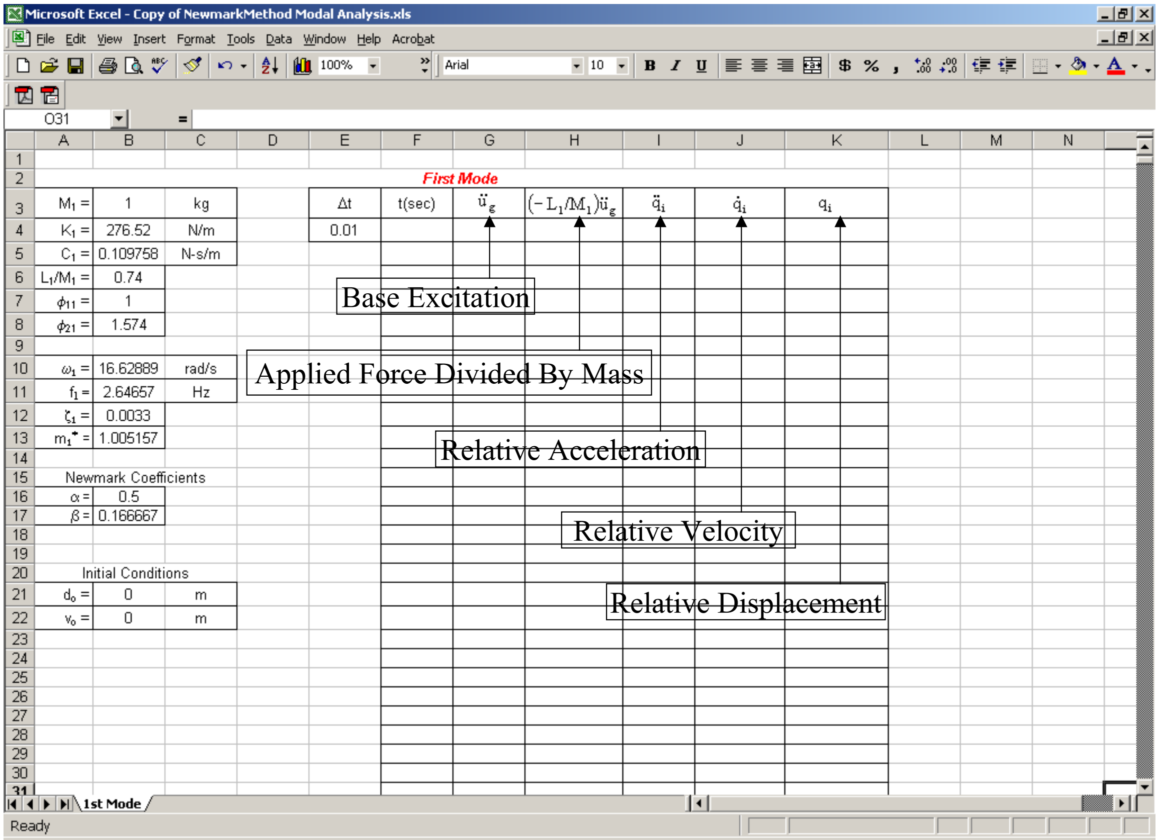


Figure 2: Spreadsheet After Completing Step 2

Place a cell (Fig. 2) for the time increment (Δt).
Place columns (Fig. 2) for the time, base excitation, applied force divided by mass, relative acceleration, relative velocity, and relative displacement.

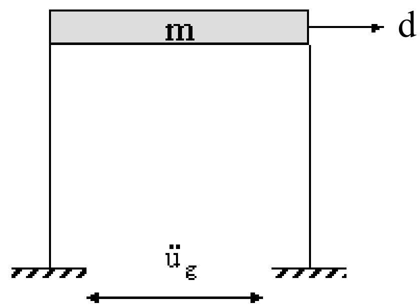
Step 3 – Enter the Time t & Applied Force f(t) into the Spreadsheet

$t_{i+1} = t_i + \Delta t$ (Equation 3) (Fig. 3)

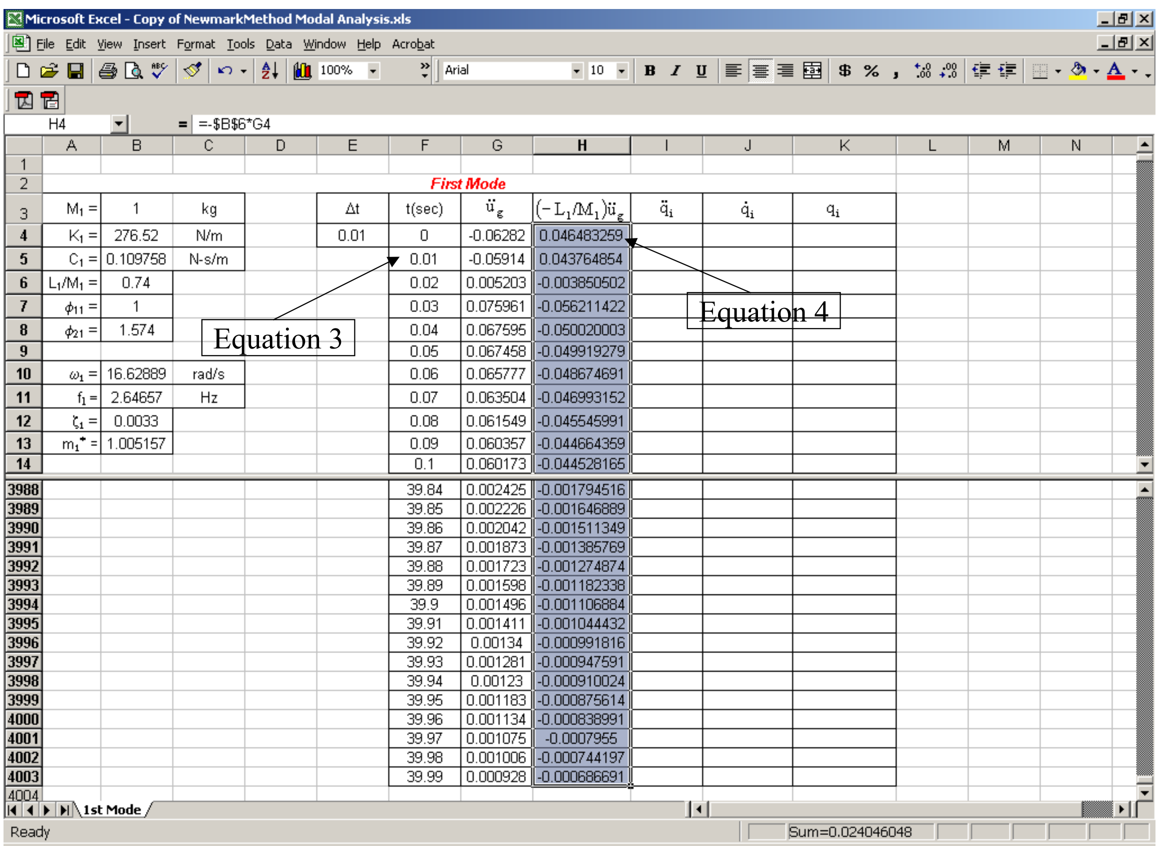
For the earthquake problem (acceleration applied to base of the structure), the applied force divided by the mass is calculated using:

$$\frac{f_i(t)}{M_i} = -\frac{L_i}{M_i}\ddot{u}_{g_i} \qquad \text{(Equation 4)} \quad \text{(Fig. 3)}$$

where, $\ddot{u}_{g_i}$ is the applied base acceleration at step i. (Typically this is the base excitation time history)



Check the units of the input motion file. They must be compatible with the units of the mass, stiffness, and damping!



	A	B	C	D	E	F	G	H	I	J	K
1											
2							*First Mode*				
3	M_1 =	1	kg		Δt	t(sec)	$\ddot{u}_g$	$(-L_1/M_1)\ddot{u}_g$	$\ddot{q}_i$	$\dot{q}_i$	q_i
4	K_1 =	276.52	N/m		0.01	0	-0.06282	0.046483259			
5	C_1 =	0.109758	N-s/m			0.01	-0.05914	0.043764854			
6	L_1/M_1 =	0.74				0.02	0.005203	-0.003850502			
7	ϕ_{11} =	1				0.03	0.075961	-0.056211422			
8	ϕ_{21} =	1.574				0.04	0.067595	-0.050020003			
9						0.05	0.067458	-0.049919279			
10	ω_1 =	16.62889	rad/s			0.06	0.065777	-0.048674691			
11	f_1 =	2.64657	Hz			0.07	0.063504	-0.046993152			
12	ζ_1 =	0.0033				0.08	0.061549	-0.045545991			
13	m_1^* =	1.005157				0.09	0.060357	-0.044664359			
14						0.1	0.060173	-0.044528165			
3988						39.84	0.002425	-0.001794516			
3989						39.85	0.002226	-0.001646889			
3990						39.86	0.002042	-0.001511349			
3991						39.87	0.001873	-0.001385769			
3992						39.88	0.001723	-0.001274874			
3993						39.89	0.001598	-0.001182338			
3994						39.9	0.001496	-0.001106884			
3995						39.91	0.001411	-0.001044432			
3996						39.92	0.00134	-0.000991816			
3997						39.93	0.001281	-0.000947591			
3998						39.94	0.00123	-0.000910024			
3999						39.95	0.001183	-0.000875614			
4000						39.96	0.001134	-0.000838991			
4001						39.97	0.001075	-0.0007955			
4002						39.98	0.001006	-0.000744197			
4003						39.99	0.000928	-0.000686691			
4004											

Figure 3: Spreadsheet After Completing Step 3

Step 4 – Compute Initial Values of the Relative Acceleration, Relative Velocity, Relative Displacement, and Absolute Acceleration

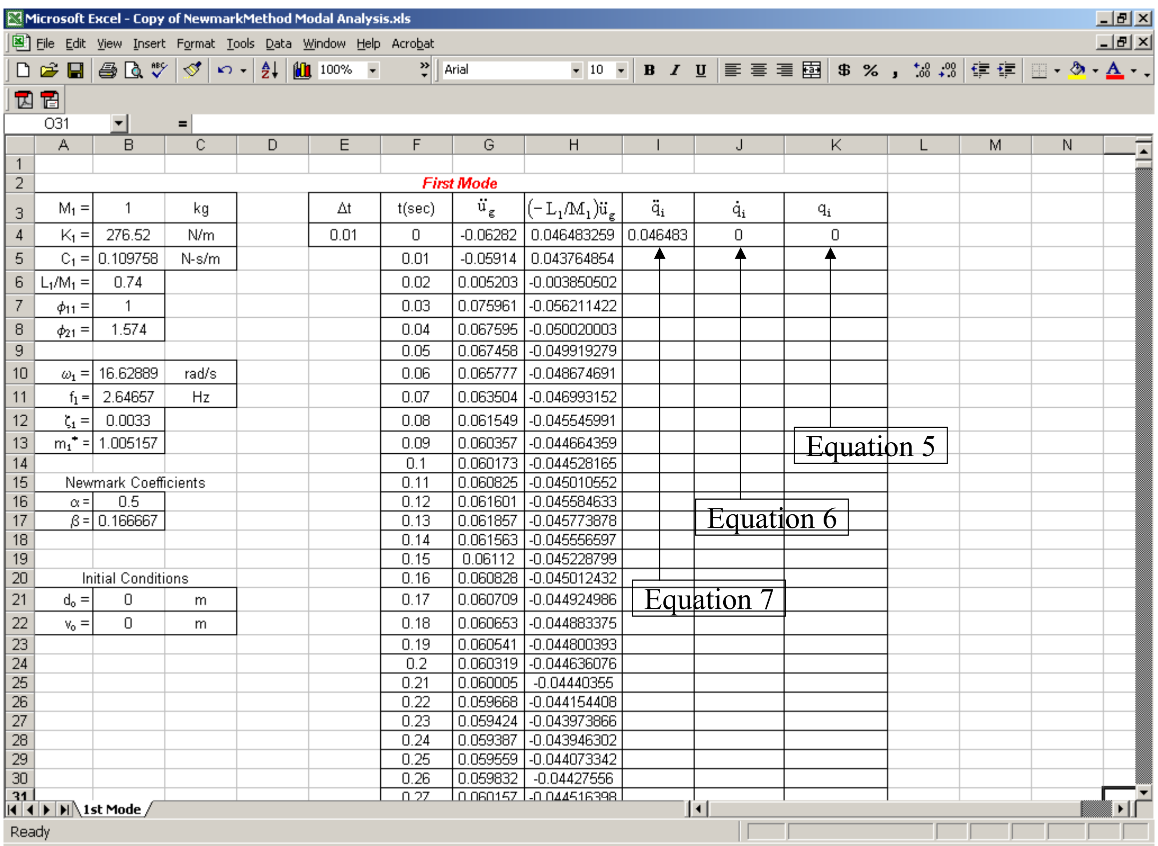
(A) The Initial Relative Displacement and Relative Velocity are known from the initial conditions (Fig. 4).

$$q(t=0) = d_o \quad \text{(Equation 5)}$$

$$\dot{q}(t=0) = v_o \quad \text{(Equation 6)}$$

(B) The Initial Relative Acceleration (Fig. 4) is calculated using

$$\ddot{q}(t=0) = -\frac{Li}{Mi}\ddot{u}_g - 2\zeta\omega v_o - \omega^2 d_o \quad \text{(Equation 7)}$$



	A	B	C	D	E	F	G	H	I	J	K
1											
2						First Mode					
3	M_1 =	1	kg		Δt	t(sec)	$\ddot{u}_g$	$(-L_1/M_1)\ddot{u}_g$	$\ddot{q}_i$	$\dot{q}_i$	q_i
4	K_1 =	276.52	N/m		0.01	0	-0.06282	0.046483259	0.046483	0	0
5	C_1 =	0.109758	N-s/m			0.01	-0.05914	0.043764854			
6	L_1/M_1 =	0.74				0.02	0.005203	-0.003850502			
7	ϕ_{11} =	1				0.03	0.075961	-0.056211422			
8	ϕ_{21} =	1.574				0.04	0.067595	-0.050020003			
9						0.05	0.067458	-0.049919279			
10	ω_1 =	16.62889	rad/s			0.06	0.065777	-0.048674691			
11	f_1 =	2.64657	Hz			0.07	0.063504	-0.046993152			
12	ζ_1 =	0.0033				0.08	0.061549	-0.045545991			
13	m_1^* =	1.005157				0.09	0.060357	-0.044664359			
14						0.1	0.060173	-0.044528165			
15	Newmark Coefficients					0.11	0.060825	-0.045010552			
16	α =	0.5				0.12	0.061601	-0.045584633			
17	β =	0.166667				0.13	0.061857	-0.045773878			
18						0.14	0.061563	-0.045556597			
19						0.15	0.06112	-0.045228799			
20	Initial Conditions					0.16	0.060828	-0.045012432			
21	d_o =	0	m			0.17	0.060709	-0.044924986			
22	v_o =	0	m			0.18	0.060653	-0.044883375			
23						0.19	0.060541	-0.044800393			
24						0.2	0.060319	-0.044636076			
25						0.21	0.060005	-0.04440355			
26						0.22	0.059668	-0.044154408			
27						0.23	0.059424	-0.043973866			
28						0.24	0.059387	-0.043946302			
29						0.25	0.059559	-0.044073342			
30						0.26	0.059832	-0.04427556			
31						0.27	0.060157	-0.044516398			

Figure 4: Spreadsheet After Completing Step 4

Step 5 – Compute Incremental Values of the Relative Acceleration, Relative Velocity, Relative Displacement, and Absolute Acceleration At Each Time Step (Fig. 5)

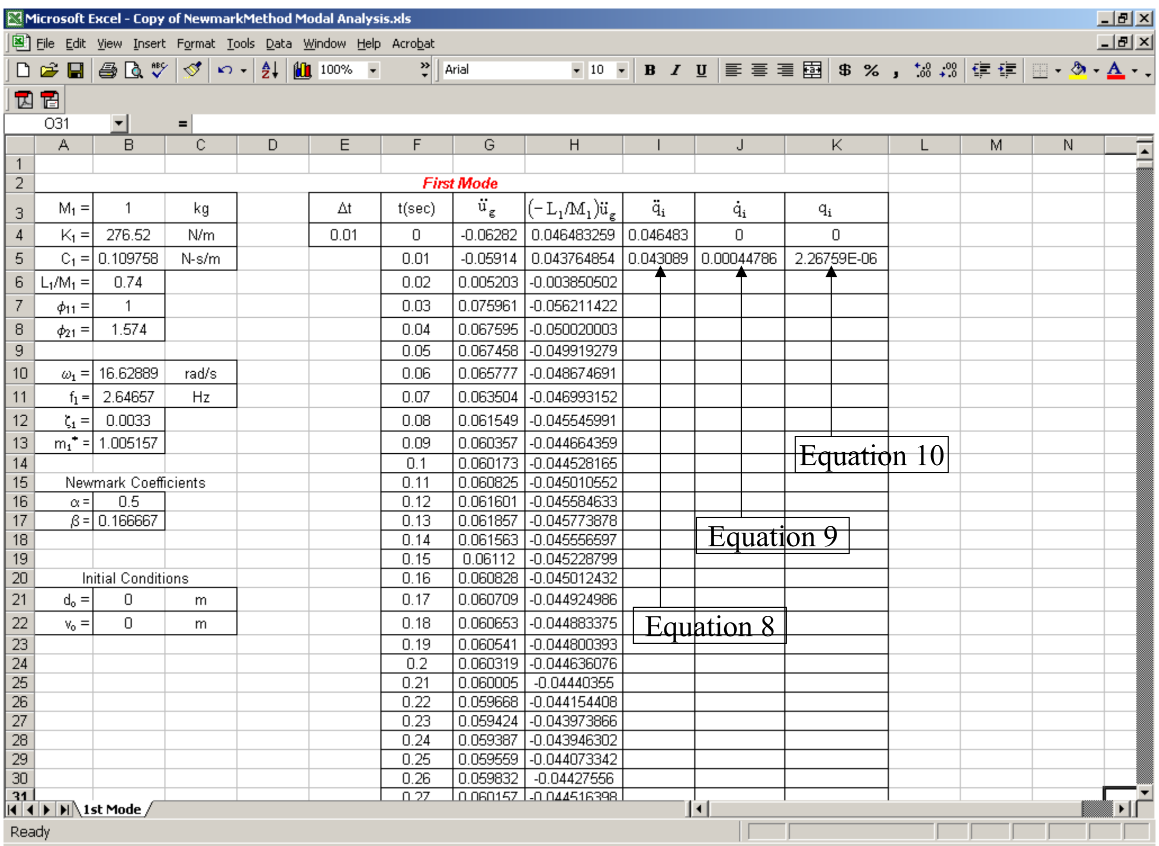
(A)

$$\ddot{q}_{i+1} = \frac{\left[-\frac{L_1}{M_1}\ddot{u}_{g\,i+1} - C_1\left(\frac{\Delta t}{2}\ddot{q}_i + \dot{q}_i\right) - K_1\left(\frac{1}{2}\Delta t^2(1-2\beta)\ddot{q}_i + \Delta t\dot{q}_i + q_i\right)\right]}{m_1{}^*} \quad \text{(Equation 8)}$$

$$\dot{q}_{i+1} = \ddot{q}_i\Delta t(1-\alpha) + \ddot{q}_{i+1}\Delta t\alpha + \dot{q}_i \quad \text{(Equation 9)}$$

$$q_{i+1} = \ddot{q}_i\frac{\Delta t^2}{2}(1-2\beta) + \ddot{q}_{i+1}\Delta t^2\beta + \dot{q}_i\Delta t + q_i \quad \text{(Equation 10)}$$

Where, the effective mass, $m_1{}^* = M_1 + C_1\Delta t\alpha + K_1\Delta t^2\beta$

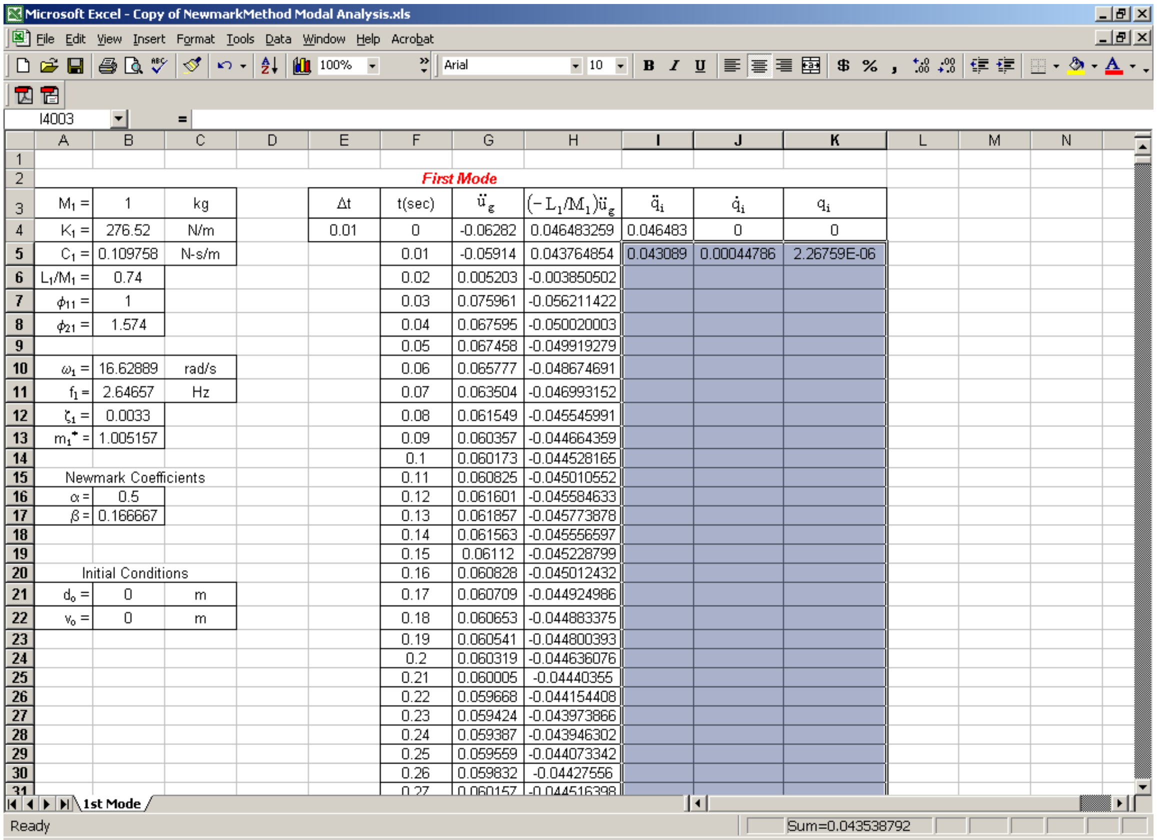


M_1 =	1	kg
K_1 =	276.52	N/m
C_1 =	0.109758	N-s/m
L_1/M_1 =	0.74	
ϕ_{11} =	1	
ϕ_{21} =	1.574	
ω_1 =	16.62889	rad/s
f_1 =	2.64657	Hz
ζ_1 =	0.0033	
$m_1{}^*$ =	1.005157	
Newmark Coefficients		
α =	0.5	
β =	0.166667	
Initial Conditions		
d_o =	0	m
v_o =	0	m

Δt	t(sec)	$\ddot{u}_g$	$(-L_1/M_1)\ddot{u}_g$	$\ddot{q}_i$	$\dot{q}_i$	q_i
0.01	0	-0.06282	0.046483259	0.046483	0	0
	0.01	-0.05914	0.043764854	0.043089	0.00044786	2.26759E-06
	0.02	0.005203	-0.003850502			
	0.03	0.075961	-0.056211422			
	0.04	0.067595	-0.050020003			
	0.05	0.067458	-0.049919279			
	0.06	0.065777	-0.048674691			
	0.07	0.063504	-0.046993152			
	0.08	0.061549	-0.045545991			
	0.09	0.060357	-0.044664359			
	0.1	0.060173	-0.044528165			
	0.11	0.060825	-0.045010552			
	0.12	0.061601	-0.045584633			
	0.13	0.061857	-0.045773878			
	0.14	0.061563	-0.045556597			
	0.15	0.06112	-0.045228799			
	0.16	0.060828	-0.045012432			
	0.17	0.060709	-0.044924986			
	0.18	0.060653	-0.044883375			
	0.19	0.060541	-0.044800393			
	0.2	0.060319	-0.044636076			
	0.21	0.060005	-0.04440355			
	0.22	0.059668	-0.044154408			
	0.23	0.059424	-0.043973866			
	0.24	0.059387	-0.043946302			
	0.25	0.059559	-0.044073342			
	0.26	0.059832	-0.04427556			
	0.27	0.060157	-0.044516398			

Figure 5: Spreadsheet with values for the Relative Acceleration, Relative Velocity, and Relative Displacement at Time Step 1

(B) Then, highlight columns I, J, & K and rows 4 through to the last time step (in this example 4003) and "Fill Down" (Ctrl+D). See Figures 6 and 7.



Microsoft Excel - Copy of NewmarkMethod Modal Analysis.xls

I4003 =

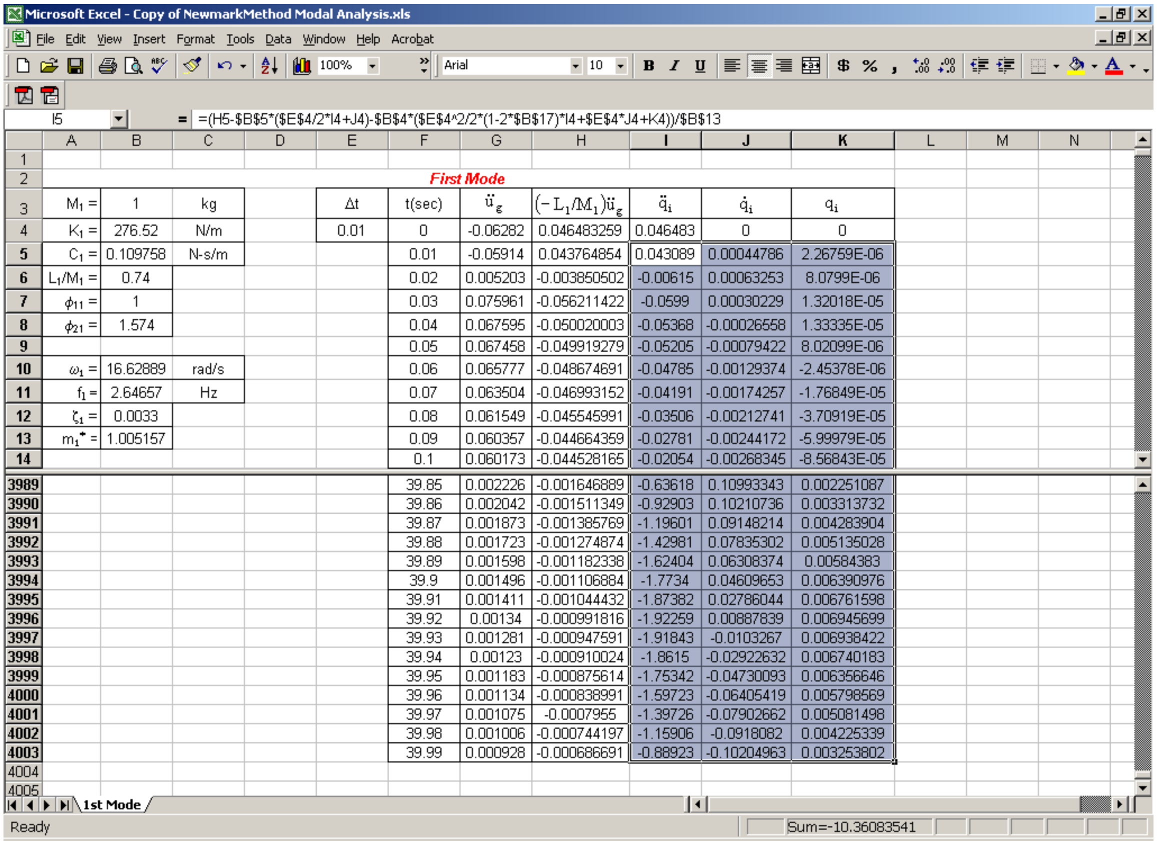
First Mode

	A	B	C	D	E	F	G	H	I	J	K
1											
2						First Mode					
3	M_1 =	1	kg		Δt	t(sec)	$\ddot{u}_g$	$(-L_1/M_1)\ddot{u}_g$	$\ddot{q}_i$	$\dot{q}_i$	q_i
4	K_1 =	276.52	N/m		0.01	0	-0.06282	0.046483259	0.046483	0	0
5	C_1 =	0.109758	N-s/m			0.01	-0.05914	0.043764854	0.043089	0.00044786	2.26759E-06
6	L_1/M_1 =	0.74				0.02	0.005203	-0.003850502			
7	ϕ_{11} =	1				0.03	0.075961	-0.056211422			
8	ϕ_{21} =	1.574				0.04	0.067595	-0.050020003			
9						0.05	0.067458	-0.049919279			
10	ω_1 =	16.62889	rad/s			0.06	0.065777	-0.048674691			
11	f_1 =	2.64657	Hz			0.07	0.063504	-0.046993152			
12	ζ_1 =	0.0033				0.08	0.061549	-0.045545991			
13	m_1^* =	1.005157				0.09	0.060357	-0.044664359			
14						0.1	0.060173	-0.044528165			
15	Newmark Coefficients					0.11	0.060825	-0.045010552			
16	α =	0.5				0.12	0.061601	-0.045584633			
17	β =	0.166667				0.13	0.061857	-0.045773878			
18						0.14	0.061563	-0.045556597			
19						0.15	0.06112	-0.045228799			
20	Initial Conditions					0.16	0.060828	-0.045012432			
21	d_o =	0	m			0.17	0.060709	-0.044924986			
22	v_o =	0	m			0.18	0.060653	-0.044883375			
23						0.19	0.060541	-0.044800393			
24						0.2	0.060319	-0.044636076			
25						0.21	0.060005	-0.04440355			
26						0.22	0.059668	-0.044154408			
27						0.23	0.059424	-0.043973866			
28						0.24	0.059387	-0.043946302			
29						0.25	0.059559	-0.044073342			
30						0.26	0.059832	-0.04427556			
31						0.27	0.060157	-0.044516398			

1st Mode

Ready Sum=0.043538792

Figure 6: Highlighted Cells



Microsoft Excel - Copy of NewmarkMethod Modal Analysis.xls

I5 = =(H5-$B$5*($E$4/2*I4+J4)-$B$4*($E$4^2/2*(1-2*$B$17)*I4+$E$4*J4+K4))/$B$13

	A	B	C	D	E	F	G	H	I	J	K
1											
2							***First Mode***				
3	M_1 =	1	kg		Δt	t(sec)	$\ddot{u}_g$	$(-L_1/M_1)\ddot{u}_g$	$\ddot{q}_i$	$\dot{q}_i$	q_i
4	K_1 =	276.52	N/m		0.01	0	-0.06282	0.046483259	0.046483	0	0
5	C_1 =	0.109758	N-s/m			0.01	-0.05914	0.043764854	0.043089	0.00044786	2.26759E-06
6	L_1/M_1 =	0.74				0.02	0.005203	-0.003850502	-0.00615	0.00063253	8.0799E-06
7	ϕ_{11} =	1				0.03	0.075961	-0.056211422	-0.0599	0.00030229	1.32018E-05
8	ϕ_{21} =	1.574				0.04	0.067595	-0.050020003	-0.05368	-0.00026558	1.33335E-05
9						0.05	0.067458	-0.049919279	-0.05205	-0.00079422	8.02099E-06
10	ω_1 =	16.62889	rad/s			0.06	0.065777	-0.048674691	-0.04785	-0.00129374	-2.45378E-06
11	f_1 =	2.64657	Hz			0.07	0.063504	-0.046993152	-0.04191	-0.00174257	-1.76849E-05
12	ζ_1 =	0.0033				0.08	0.061549	-0.045545991	-0.03506	-0.00212741	-3.70919E-05
13	m_1^* =	1.005157				0.09	0.060357	-0.044664359	-0.02781	-0.00244172	-5.99979E-05
14						0.1	0.060173	-0.044528165	-0.02054	-0.00268345	-8.56843E-05
3989						39.85	0.002226	-0.001646889	-0.63618	0.10993343	0.002251087
3990						39.86	0.002042	-0.001511349	-0.92903	0.10210736	0.003313732
3991						39.87	0.001873	-0.001385769	-1.19601	0.09148214	0.004283904
3992						39.88	0.001723	-0.001274874	-1.42981	0.07835302	0.005135028
3993						39.89	0.001598	-0.001182338	-1.62404	0.06308374	0.00584383
3994						39.9	0.001496	-0.001106884	-1.7734	0.04609653	0.006390976
3995						39.91	0.001411	-0.001044432	-1.87382	0.02786044	0.006761598
3996						39.92	0.00134	-0.000991816	-1.92259	0.00887839	0.006945699
3997						39.93	0.001281	-0.000947591	-1.91843	-0.0103267	0.006938422
3998						39.94	0.00123	-0.000910024	-1.8615	-0.02922632	0.006740183
3999						39.95	0.001183	-0.000875614	-1.75342	-0.04730093	0.006356646
4000						39.96	0.001134	-0.000838991	-1.59723	-0.06405419	0.005798569
4001						39.97	0.001075	-0.0007955	-1.39726	-0.07902662	0.005081498
4002						39.98	0.001006	-0.000744197	-1.15906	-0.0918082	0.004225339
4003						39.99	0.000928	-0.000686691	-0.88923	-0.10204963	0.003253802
4004											
4005											

1st Mode

Ready   Sum=-10.36083541

Figure 7: Spreadsheet After “Filling Down” Columns I through K

Step 6 – Create Additional Worksheet for Second Mode

Make a copy of the "1st Mode" worksheet by right clicking on the "1st Mode" tab and selecting "Move or Copy" (Fig. 8)

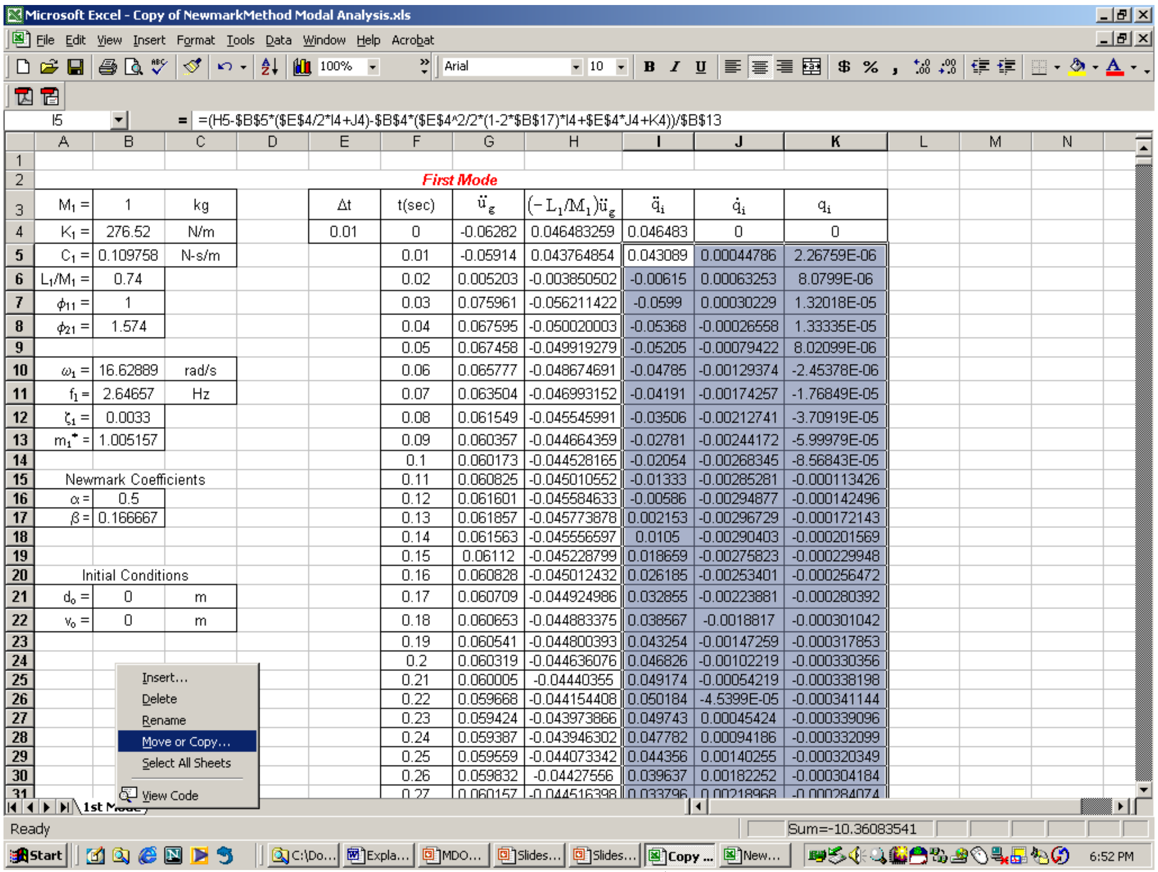


Figure 8: Creating a Copy of 1st Mode Worksheet

Then check the box for "Create a copy" and click on "OK" button (Fig. 9)

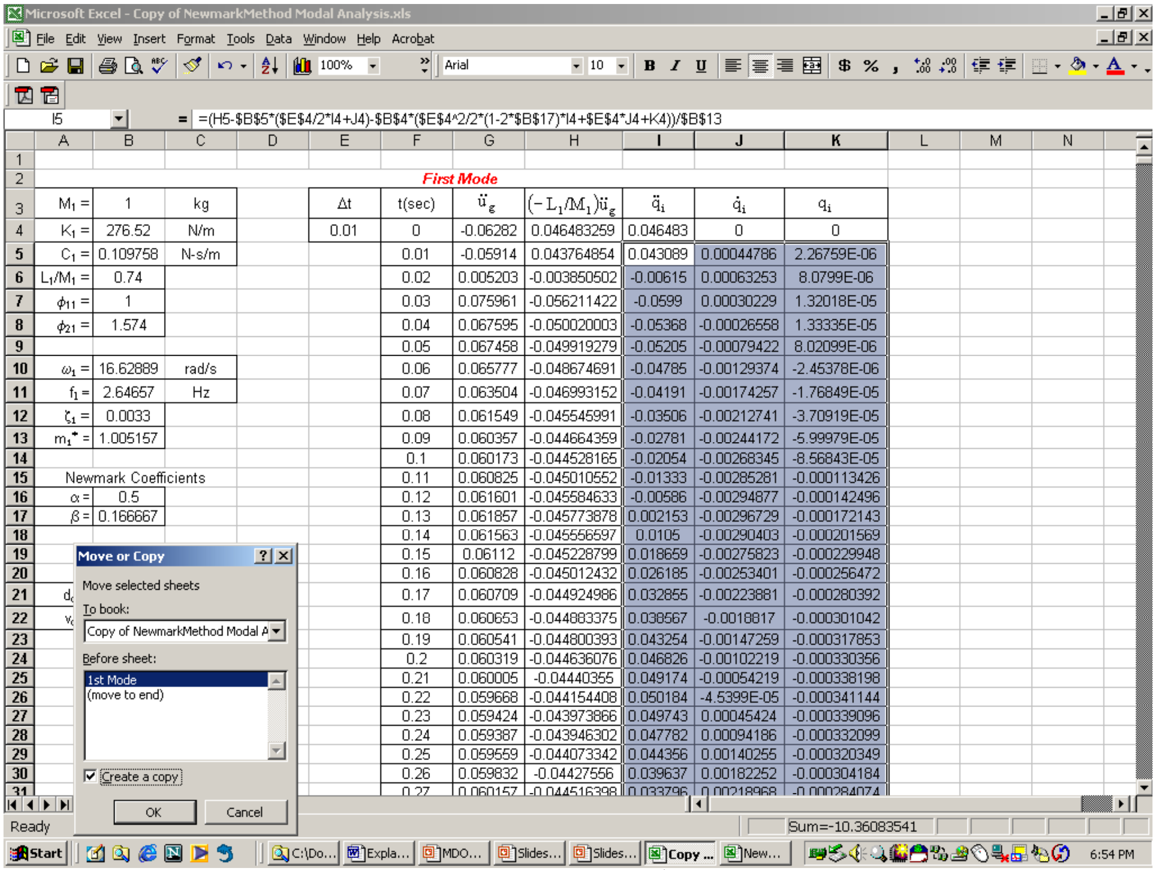


Figure 9: Creating a Copy of 1st Mode Worksheet

Rename this worksheet by right clicking on the "1st Mode (2)" tab and selecting "Rename". Rename this worksheet "2nd Mode" (Fig. 10)

Enter the appropriate values for M_2, K_2, C_2, $\frac{L_2}{M_2}$, ϕ_2, d_o, and v_o (Fig. 10).

Microsoft Excel - NewmarkMethod Modal Analysis.xls

Second Mode

M_2 =	1	kg
K_2 =	1951.652	N/m
C_2 =	0.15152	N-s/m
L_2/M_2 =	0.26	
ϕ_{12} =	1	
ϕ_{22} =	-0.6356	
ω_2 =	44.17751	rad/s
f_2 =	7.031068	Hz
ζ_1 =	0.001715	
m_2^* =	1.033285	
Newmark Coefficients		
α =	0.5	
β =	0.166667	
Initial Conditions		
d_o =	0	m
v_o =	0	m

Δt	t(sec)	$\ddot{u}_g$	$(-L_1/M_1)\ddot{u}_g$	$\ddot{q}_i$	$\dot{q}_i$	q_i
0.01	0	-0.06282	0.016331956	0.016332	0	0
	0.01	-0.05914	0.01537684	0.013841	0.00015087	7.75087E-07
	0.02	0.005203	-0.001352879	-0.00653	0.00018744	2.63635E-06
	0.03	0.075961	-0.019749959	-0.02725	1.8581E-05	3.83911E-06
	0.04	0.067595	-0.017574596	-0.02288	-0.00023204	2.73543E-06
	0.05	0.067458	-0.017539206	-0.01627	-0.00042776	-6.1865E-07
	0.06	0.065777	-0.017101918	-0.0062	-0.00054012	-5.54191E-06
	0.07	0.063504	-0.016511108	0.005164	-0.00054532	-1.10638E-05
	0.08	0.061549	-0.016002645	0.015461	-0.00044219	-1.60872E-05
	0.09	0.060357	-0.015692883	0.02263	-0.00025174	-1.96166E-05
	0.1	0.060173	-0.015645031	0.025261	-1.2284E-05	-2.09586E-05
	0.11	0.060825	-0.015814518	0.022906	0.00022855	-1.98576E-05
	0.12	0.061601	-0.016016222	0.016197	0.00042407	-1.65387E-05
	0.13	0.061857	-0.016082714	0.00657	0.0005379	-1.16486E-05
	0.14	0.061563	-0.016006372	-0.00415	0.00055002	-6.11969E-06
	0.15	0.06112	-0.0158912	-0.01403	0.00045916	-9.91467E-07
	0.16	0.060828	-0.015815179	-0.02128	0.00028263	2.77794E-06
	0.17	0.060709	-0.015784454	-0.02455	5.3498E-05	4.48582E-06
	0.18	0.060653	-0.015769834	-0.02319	-0.00018519	3.81607E-06
	0.19	0.060541	-0.015740678	-0.01744	-0.00038833	9.00572E-07
	0.2	0.060319	-0.015682946	-0.00838	-0.00051741	-3.70365E-06
	0.21	0.060005	-0.015601247	0.002279	-0.0005479	-9.11897E-06
	0.22	0.059668	-0.015513711	0.012494	-0.00047403	-1.43137E-05
	0.23	0.059424	-0.015450277	0.02031	-0.00031002	-1.82991E-05
	0.24	0.059387	-0.015440593	0.024227	-8.7328E-05	-2.03185E-05
	0.25	0.059559	-0.015485228	0.02351	0.00015136	-1.99924E-05
	0.26	0.059832	-0.015556278	0.018328	0.00036055	-1.73897E-05
	0.27	0.060157	-0.015640896	0.009678	0.00050057	-1.3012E-05

1st Mode | 2nd Mode | 1st Floor | 2nd Floor | q1 | q2 | a1 | v1 | d1 | a1T | a2 | v2 | d2 | a2T

Ready

Figure 10: Worksheet for Second Mode

Step 7 – Repeat Step 6 for Additional Modes

Step 8 – Determine the Response at Each of the Floors

Determine the Response of the first floor using the equations:

$$\mathbf{u} = \mathbf{\Phi q}$$
$$\dot{\mathbf{u}} = \mathbf{\Phi}\dot{\mathbf{q}}$$
$$\ddot{\mathbf{u}} = \mathbf{\Phi}\ddot{\mathbf{q}}$$

For example for a 2DOF structure, the first floor response is

$$u_1 = \phi_{11} q_1 + \phi_{12} q_2 \qquad \text{(Equation 11)}$$
$$\dot{u}_1 = \phi_{11} \dot{q}_1 + \phi_{12} \dot{q}_2 \qquad \text{(Equation 12)}$$
$$\ddot{u}_1 = \phi_{11} \ddot{q}_1 + \phi_{12} \ddot{q}_2 \qquad \text{(Equation 13)}$$

and the second floor response is

$$u_2 = \phi_{21} q_1 + \phi_{22} q_2 \qquad \text{(Equation 14)}$$

$$\dot{u}_2 = \phi_{21}\dot{q}_1 + \phi_{22}\dot{q}_2 \qquad \text{(Equation 15)}$$

$$\ddot{u}_2 = \phi_{21}\ddot{q}_1 + \phi_{22}\ddot{q}_2 \qquad \text{(Equation 16)}$$

The first floor absolute acceleration is $\ddot{u}_1^T = \ddot{u}_1 + \ddot{u}_g$ (Equation 17)

The second floor absolute acceleration is $\ddot{u}_2^T = \ddot{u}_2 + \ddot{u}_g$ (Equation 18)