Systems with Distributed Mass and Elasticity

1. Free vibration of a bending beam

Bending Beam Equation:

$$m\ddot{u} + EIu'''' = 0 \tag{1}$$

Define solution as:

$$u(x,t) = \phi(x)q(t) \tag{2}$$

Substitute Steady state

$$\ddot{u} = -\omega^2 u \tag{3}$$

$$EI\phi'''' - \omega^2 m\phi = 0 \tag{4}$$

$$EI\phi'''' - \beta^4 m\phi = 0 \tag{5}$$

$$\beta^4 = (\omega^2 m / EI) \tag{6}$$

Solution is:

$$\phi(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \tag{7}$$

For a simply supported beam, the boundary conditions are:

$$u(0) = 0, u''(0) = 0, u(L) = 0, u''(L) = 0 \tag{8}$$

Leading to:

$$\sin \beta L = 0 \tag{9}$$

or

$$\beta L = n\pi \tag{10}$$

or:

$$w_n = (n^2\pi^2 / L^2)(\sqrt{EI/M}) \tag{11}$$

and:

$$\phi_n(x) = C_1 \sin(n\pi x / L) \tag{12}$$

For a cantilever beam:

$$u(0) = 0, u'(0) = 0, u''(L) = 0, u'''(L) = 0 \tag{13}$$

Leading to:

$$1 + \cos\beta L \cosh\beta L = 0 \tag{14}$$

or:

$$\begin{aligned} w_1 &= (3.516 / L^2)(\sqrt{EI/M}), w_2 = (22.03 / L^2)(\sqrt{EI/M}), \\ w_3 &= (61.70 / L^2)(\sqrt{EI/M}), w_4 = (120.9 / L^2)(\sqrt{EI/M}) \end{aligned} \tag{15}$$

2. Modal Orthogonality

Steady state response of an SDOF system to periodic excitation is obtained at the aid of Fourier Transform and Inverse Fourier Transform.

For a cantilever beam:

$$EI\phi_n'''' = \omega_n^2 m\phi_n \tag{16}$$

Leading to:

$$\int_o^L \phi_r EI\phi_n'''' dx = \omega_n^2 \int_o^L \phi_r m\phi_n dx \tag{17}$$

Leading to:

$$\int_o^L EI\phi_r''\phi_n'' dx = \omega_n^2 \int_o^L m\phi_r\phi_n dx \tag{18}$$

Replacing everything above but starting with mode r and multiplying by mode n:

$$\int_o^L EI\phi_r''\phi_n'' dx = \omega_r^2 \int_o^L m\phi_r\phi_n dx \tag{19}$$

Subtracting:

$$(\omega_r^2 - \omega_n^2)\int_o^L m\phi_r\phi_n dx = 0 \tag{20}$$

for:

$$(\omega_r \neq \omega_n) \tag{21}$$

The orthogonality condition becomes:

$$\int_o^L m\phi_r\phi_n dx = 0 \tag{22}$$

or:

$$\int_o^L EI\phi_r''\phi_n'' dx = 0 \tag{23}$$

Modal Analysis

Define solution as:

$$u(x,t) = \sum_{r=1}^{\infty} \phi_r(x) q_r(t) \tag{24}$$

Substituting in beam equation:

$$\sum_{r=1}^{\infty} m\phi_r \ddot{q}_r + \sum_{r=1}^{\infty} EI\phi_r'''' q_r = -m\ddot{u}_g \tag{25}$$

Multiply by ϕ_n and integrate over domain *0-L*:

$$\int_0^L \phi_n \sum_{r=1}^{\infty} m\phi_r \ddot{q}_r dx + \int_0^L \phi_n \sum_{r=1}^{\infty} EI\phi_r'''' q_r dx = -\int_0^L \phi_n m dx \ddot{u}_g \tag{26}$$

Due to modal orthogonality:

$$\int_0^L m\phi_n^{\,2} dx \ddot{q}_n + \int_0^L \phi_n EI\phi_n'''' q_n dx = -\int_0^L \phi_n m dx \ddot{u}_g \tag{27}$$

or:

$$M_n \ddot{q}_n + K_n q_n = -L_n \ddot{u}_g \tag{28}$$

and as always,

$$K_n = \omega^2 M_n \tag{29}$$

Now we can solve each equation for qn independently, and then calculate u:

or:

$$\ddot{q}_n + 2\zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = -(L_n / M_n)\ddot{u}_g \tag{30}$$

and,

$$u(x,t) = \sum_{r=1}^{\infty} \phi_r(x) q_r(t) \tag{31}$$

Homework

1) Find mode shapes and natural frequencies of a cantilever shear beam defined by:

$$\rho\ddot{u} = Gu'' \tag{3}$$

And boundary conditions:

$$u(0) = 0.......Gu'(h) = 0 \tag{4}$$

Where ρ is mass density, G is shear modulus, and h is beam height.

Hint: Note that the shear wave velocity V_s is defined by:

$$V_s = \sqrt{G/\rho} \tag{5}$$

and the natural frequencies in Hertz are:

$$f_n = (2n-1)V_s/4h.......n = 1,2,3,.. \tag{6}$$

2) Use a finite element program (bending beam elements) to model a 30 m cantilever bending beam ((choose EI to represent a building), and subject it to a unit load at the top. Compare the result from 3 different meshes of 2 elements, 6 elements and 10 elements. Repeat specifying a zero rotation at the top. Compare the results to the theoretical deflection at the beam top (both cases, free at the top, and zero rotation at the top).

3) Use a Finite Element program (Beam Elements) to model a cantilever bending beam 30 m in height (choose EI and m to represent a building). Build 3 meshes, of 2 elements, 6 elements and 10 elements, and compute mode shapes and natural frequencies in each case (focus on first 4 mode shapes and frequencies). Plot the mode shapes in each case. Compare the resonant

frequency results and verify based on the equations for natural frequency provided in this handout.